

SYMMETRIES OF FLAT RANK TWO DISTRIBUTIONS AND SUB-RIEMANNIAN STRUCTURES

YURI L. SACHKOV

ABSTRACT. Flat sub-Riemannian structures are local approximations — nilpotentizations — of sub-Riemannian structures at regular points. Lie algebras of symmetries of flat maximal growth distributions and sub-Riemannian structures of rank two are computed in dimensions 3, 4, and 5.

1. SUB-RIEMANNIAN STRUCTURES

A *sub-Riemannian geometry* is a triple $(M, \Delta, \langle \cdot, \cdot \rangle)$, where M is a smooth manifold, $\Delta \subset TM$ is a smooth distribution on M , $\Delta = \{\Delta_q \subset T_q M \mid q \in M\}$, and $\langle \cdot, \cdot \rangle$ is an inner product in Δ that smoothly depends on a point in M , $\langle \cdot, \cdot \rangle = \{\langle \cdot, \cdot \rangle_q \mid q \in M\}$ — an inner product in $\Delta_q \mid q \in M\}$. The pair $(\Delta, \langle \cdot, \cdot \rangle)$ is a *sub-Riemannian structure* on M ; if $\dim M = n$ and $\dim \Delta_q = k$, $q \in M$, then we say that $(\Delta, \langle \cdot, \cdot \rangle)$ is a (k, n) -*structure*. The number k is called the *rank* of the distribution Δ or the structure $(\Delta, \langle \cdot, \cdot \rangle)$.

In this work, we are interested in the special class of distributions and sub-Riemannian structures called *flat*. Let G be a connected simply connected nilpotent Lie group. Suppose that its Lie algebra $\mathfrak{g}$ is graded:

$$\begin{aligned}\mathfrak{g} &= \mathfrak{g}^1 \oplus \mathfrak{g}^2 \oplus \cdots \oplus \mathfrak{g}^s, \\ [\mathfrak{g}^i, \mathfrak{g}^j] &\subset \mathfrak{g}^{i+j}, \quad \mathfrak{g}^r = \{0\} \quad \forall r > s,\end{aligned}$$

and generated as a Lie algebra by its component of degree 1:

$$\text{Lie}(\mathfrak{g}^1) = \mathfrak{g}.$$

Then

$$\Delta = \mathfrak{g}^1$$

can be considered as a completely nonintegrable (bracket-generating) left-invariant distribution on the Lie group G . We call such a distribution Δ *flat*. Further, if Δ is equipped with a left-invariant inner product $\langle \cdot, \cdot \rangle$ obtained from an inner product in $\mathfrak{g}^1$, then $(\Delta, \langle \cdot, \cdot \rangle)$ is called a *flat* sub-Riemannian structure on G . Flat sub-Riemannian structures arise as local approximations — nilpotentizations — of arbitrary sub-Riemannian structures at regular points (see [2], [3] for details).

Received by the editors May 4, 2001.

2000 *Mathematics Subject Classification*. Primary 53C17.

Key words and phrases. Sub-Riemannian geometry, symmetries, distributions, sub-Riemannian structures.

This work was partially supported by the Russian Foundation for Basic Research, project No. 02-01-00506.

For a distribution $\Delta \subset TM$ its *Lie flag* is defined as follows:

$$\Delta \subset \Delta^2 = \Delta + [\Delta, \Delta] \subset \Delta^3 = \Delta^2 + [\Delta, \Delta^2] \subset \cdots \subset TM$$

(here Δ denotes also the $C^\infty(M)$ -module of vector fields on M tangent to the distribution Δ). Then the *growth vector* of Δ at a point $q \in M$ is the vector

$$(n_1, n_2, n_3, \dots), \quad n_i = \dim \Delta^i(q).$$

For a flat distribution $\Delta \subset TG$ we may restrict ourselves to left-invariant vector fields on the Lie group G :

$$\Delta^i(q) = (\mathfrak{g}^1 \oplus \cdots \oplus \mathfrak{g}^i)(q),$$

and the growth vector is constant and takes the form

$$(n_1, n_2, n_3, \dots), \quad n_i = \sum_{j=1}^i \dim \mathfrak{g}^j.$$

Two flat distributions (sub-Riemannian structures) are called *isomorphic* if there exists an isomorphism of Lie algebras that maps the first distribution (respectively, sub-Riemannian structure) onto the second one, in other words, if they are isomorphic as left-invariant objects on G .

The aim of this work is to study symmetries of flat distributions and sub-Riemannian structures in dimensions $(2, n)$, $n = 3, 4, 5$, for maximal growth vectors. More precisely, we consider the following cases:

Dimension	Growth vector
(2, 3)	(2, 3)
(2, 4)	(2, 3, 4)
(2, 5)	(2, 3, 5)

Of course, the maximum growth condition is essential only for dimension $(2, 5)$ since for dimensions $(2, 3)$ and $(2, 4)$ the growth vectors are uniquely determined.

We are interested in the maximum growth case since it is generic: a generic distribution has the maximum possible growth at a generic point. The case $(2, 3, 5)$ is important for applications: sub-Riemannian structures with the growth vector $(2, 3, 5)$ appear in the following systems:

1) a pair of bodies rolling one on another without slipping or twisting [1], [10]; in particular, the sphere rolling on a plane, the plate-ball problem [8];

2) a car with 2 off-hooked trailers [9], [12].

In conclusion, we notice that the results of this work on symmetries of flat distributions are not new. In particular, symmetries of the flat $(2, 3, 5)$ distribution were known to E. Cartan [6]. However, this result does not seem to be presented in the modern terminology elsewhere.

The results on symmetries of flat sub-Riemannian structures are new.

2. SYMMETRIES

In this work, the term “smooth” means C^∞ . Given a smooth vector field $X \in \text{Vec}(M)$, we denote by e^{tX} its flow, by $(e^{tX})_*$ the differential (push-forward) action of the flow on vector fields, and by $(e^{tX})^*$ the pull-back action of the flow on forms.

A vector field $X \in \text{Vec}(M)$ is called an (infinitesimal) *symmetry*:

1) of a distribution Δ on M if its flow preserves Δ :

$$(e^{tX})_*\Delta = \Delta, \quad t \in \mathbb{R};$$

2) of a sub-Riemannian structure $(\Delta, \langle \cdot, \cdot \rangle)$ on M if its flow preserves both Δ and $\langle \cdot, \cdot \rangle$:

$$(e^{tX})_*\Delta = \Delta, \quad (e^{tX})^*\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle, \quad t \in \mathbb{R}.$$

The Lie algebras of symmetries of a distribution Δ , or a sub-Riemannian structure $(\Delta, \langle \cdot, \cdot \rangle)$, will be denoted by $\text{Sym}(\Delta)$, respectively $\text{Sym}(\Delta, \langle \cdot, \cdot \rangle)$.

Any left-invariant object on a Lie group G (e.g., a vector field, a distribution, or a sub-Riemannian structure) is, by definition, preserved by left translations on G . On the other hand, the flow of a right-invariant vector field X on G acts as a left translation:

$$e^{tX}(g) = e^{tX}(\text{Id})g, \quad g \in G, \quad t \in \mathbb{R},$$

where Id is the identity element of G . So any right-invariant vector field is an infinitesimal symmetry of any left-invariant object. In particular, for any left-invariant distribution Δ and left-invariant sub-Riemannian structure $(\Delta, \langle \cdot, \cdot \rangle)$ we have

$$(2.1) \quad \mathfrak{g}_r \subset \text{Sym}(\Delta, \langle \cdot, \cdot \rangle) \subset \text{Sym}(\Delta).$$

Here $\mathfrak{g}_r$ is the Lie algebra of right-invariant vector fields on G , which is isomorphic to the Lie algebra $\mathfrak{g}$ of G .

Symmetries of distributions and sub-Riemannian structures can be computed via the following statement.

Proposition 1. *Let $X \in \text{Vec}(M)$.*

(1) $X \in \text{Sym}(\Delta)$ *iff* $\text{ad } X(\Delta) \subset \Delta$, *or, equivalently,*

$$(2.2) \quad \text{ad } X \in \mathfrak{gl}(\Delta).$$

(2) $X \in \text{Sym}(\Delta, \langle \cdot, \cdot \rangle)$ *iff*

$$(2.3) \quad \text{ad } X \in \mathfrak{so}(\Delta, \langle \cdot, \cdot \rangle).$$

Any right-invariant vector field X on a Lie group G commutes with any left-invariant vector field; thus,

$$\text{ad } X|_{\Delta} = 0$$

for a left-invariant distribution Δ on G . This gives another proof of inclusion (2.1).

Remark 1. Inclusion (2.2) means that for any vector field $\xi \in \text{Vec}(M)$ tangent to the distribution Δ , the Lie bracket $[X, \xi]$ is tangent to Δ as well:

$$\xi \in \Delta \quad \Rightarrow \quad [X, \xi] \in \Delta.$$

That is, there is defined a linear mapping

$$(2.4) \quad \text{ad } X : \Delta \rightarrow \Delta, \quad \text{ad } X : \xi \mapsto [X, \xi].$$

In terms of a local basis, condition (2.2) reads as follows. For any point $q_0 \in M$ and any local basis $\xi_1, \dots, \xi_k$ of the distribution Δ in a neighborhood $q_0 \in O \subset M$:

$$\Delta_q = \text{span}(\xi_1(q), \dots, \xi_k(q)), \quad q \in O,$$

there exist smooth functions $c_{ij} = c_{ij}(q)$ defined on O such that

$$(2.5) \quad [X, \xi_i](q) = \sum_{j=1}^k c_{ji}(q) \xi_j(q), \quad q \in O, \quad i = 1, \dots, k.$$

Similarly, inclusion (2.3) means that the linear mapping (2.4) is skew-symmetric with respect to the inner product $\langle \cdot, \cdot \rangle$. In terms of a local basis: for any point $q_0 \in M$ and any local orthonormal basis of the sub-Riemannian structure $(\Delta, \langle \cdot, \cdot \rangle)$ in a neighborhood O of q_0 ,

$$\begin{aligned} \Delta_q &= \text{span}(\xi_1(q), \dots, \xi_k(q)), \\ \langle \xi_i(q), \xi_j(q) \rangle &= \delta_{ij}, \quad q \in O, \quad i, j = 1, \dots, k, \end{aligned}$$

equality (2.5) is satisfied for some smooth functions $c_{ij} = c_{ij}(q)$ defined on O such that the matrix $C = C(q) = (c_{ij})_{i,j=1}^k$ is skew-symmetric:

$$C^* = -C, \quad q \in O.$$

The preceding equality is equivalent to the following one:

$$\langle [X, \xi_i], \xi_j \rangle + \langle \xi_i, [X, \xi_j] \rangle = 0, \quad i, j = 1, \dots, k.$$

Now we prove Proposition 1.

Proof. Statement (1) is well known; see, e.g., Theorem 3.1 [4]. We prove statement (2).

Let $\xi_1, \dots, \xi_k$ and $\eta_1, \dots, \eta_k$ be local orthonormal bases of the sub-Riemannian structure $(\Delta, \langle \cdot, \cdot \rangle)$ near points $q \in M$ and $q_t = e^{tX}(q)$ respectively. Fix any pair of indices $i, j \in \{1, \dots, k\}$ and define a smooth function depending on a parameter t :

$$\varphi_t = \langle (e^{tX})_* \xi_i, (e^{tX})_* \xi_j \rangle.$$

Necessity. Let $X \in \text{Sym}(\Delta, \langle \cdot, \cdot \rangle)$. Then

$$\varphi_t \equiv \varphi_0 = \delta_{ij};$$

thus,

$$0 = \left. \frac{d}{dt} \right|_{t=0} \varphi_t = \langle -\text{ad } X(\xi_i), \xi_j \rangle + \langle \xi_i, -\text{ad } X(\xi_j) \rangle.$$

The equalities

$$\langle \text{ad } X(\xi_i), \xi_j \rangle + \langle \xi_i, \text{ad } X(\xi_j) \rangle = 0, \quad i, j = 1, \dots, k,$$

mean that the operator $\text{ad } X : \Delta \rightarrow \Delta$ is skew-symmetric with respect to the inner product $\langle \cdot, \cdot \rangle$.

Sufficiency. Let $\text{ad } X \in \text{so}(\Delta, \langle \cdot, \cdot \rangle)$. In particular, inclusion (2.2) is satisfied. By item (1) of this proposition, the flow e^{tX} preserves the distribution Δ , i.e.,

$$(e^{tX})_* \xi_i = \sum_{m=1}^k a_{mi} \eta_m, \quad i = 1, \dots, k,$$

for some smooth functions a_{mi} defined in a neighborhood of the point q_t . Then

$$(2.6) \quad \varphi_t = \left\langle \sum_{m=1}^k a_{mi} \eta_m, \sum_{l=1}^k a_{lj} \eta_l \right\rangle = \sum_{m,l=1}^k a_{mi} a_{lj} \underbrace{\langle \eta_m, \eta_l \rangle}_{=\delta_{ml}} = \sum_{l=1}^k a_{li} a_{lj}.$$

On the other hand,

$$\begin{aligned}
\frac{d\varphi_t}{dt} &= \langle -\operatorname{ad} X \circ (e^{tX})_* \xi_i, (e^{tX})_* \xi_j \rangle + \langle (e^{tX})_* \xi_i, -\operatorname{ad} X \circ (e^{tX})_* \xi_j \rangle \\
&= -\left\langle \operatorname{ad} X \left(\sum_{m=1}^k a_{mi} \eta_m \right), \sum_{l=1}^k a_{lj} \eta_l \right\rangle \\
&\quad - \left\langle \sum_{m=1}^k a_{mi} \eta_m, \operatorname{ad} X \left(\sum_{l=1}^k a_{lj} \eta_l \right) \right\rangle \\
&= -\sum_{m,l=1}^k \left(((X a_{mi}) a_{lj} + a_{mi} (X a_{lj})) \underbrace{\langle \eta_m, \eta_l \rangle}_{=\delta_{ij}} \right. \\
&\quad \left. + a_{mi} a_{lj} \underbrace{(\langle (\operatorname{ad} X) \eta_m, \eta_l \rangle + \langle \eta_m, (\operatorname{ad} X) \eta_l \rangle)}_{=0} \right) \\
&= -\sum_{l=1}^k (X a_{li}) a_{lj} + a_{li} (X a_{lj}) \\
&= -X \left(\sum_{l=1}^k a_{li} a_{lj} \right).
\end{aligned}$$

Here Xf denotes the Lie derivative (directional derivative) of the function f along the vector field X :

$$Xf = df(X), \quad f \in C^\infty(M), \quad X \in \operatorname{Vec}(M).$$

In view of equality (2.6), we obtain that the family of functions φ_t is a solution of the ODE

$$(2.7) \quad \frac{d\varphi_t}{dt} = -X\varphi_t.$$

It is easy to see that this ODE has a unique solution: if φ_t satisfies (2.7), then

$$\frac{d}{dt} \varphi_t(e^{tX} q) = -X\varphi_t(e^{tX} q) + X\varphi_t(e^{tX} q) = 0;$$

thus,

$$\varphi_t(e^{tX} q) \equiv \varphi_0(q),$$

i.e.,

$$\varphi_t(q) = \varphi_0(e^{-tX} q).$$

But $\varphi_0 \equiv \delta_{ij}$; thus,

$$\varphi_t = \langle (e^{tX})_* \xi_i, (e^{tX})_* \xi_j \rangle \equiv \delta_{ij}, \quad i, j = 1, \dots, k,$$

i.e., the field X is an infinitesimal symmetry of the sub-Riemannian structure $(\Delta, \langle \cdot, \cdot \rangle)$. $\square$

3. THE HEISENBERG CASE

3.1. The flat distribution and sub-Riemannian structure. Let $\mathfrak{g}$ be the three-dimensional Heisenberg algebra, i.e., the (unique) three-dimensional two-step nilpotent Lie algebra:

$$(3.1) \quad \dim \mathfrak{g} = 3, \quad \dim[\mathfrak{g}, \mathfrak{g}] = 1, \quad \dim[\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]] = 0,$$

and let G be the three-dimensional Heisenberg group, i.e., the corresponding connected simply connected Lie group. A flat rank two distribution Δ on G is just any rank two nonintegrable left-invariant distribution on G :

$$\Delta \subset \mathfrak{g}, \quad \dim \Delta = 2, \quad \text{Lie}(\Delta) = \mathfrak{g}.$$

To obtain a flat sub-Riemannian structure on G one has to add any left-invariant inner product $\langle \cdot, \cdot \rangle$ in Δ .

As was indicated in [13], up to isomorphism there exists exactly one flat distribution on the Heisenberg group, and the same is true for flat sub-Riemannian structures. To show this, choose an orthonormal frame:

$$(3.2) \quad \Delta = \text{span}(\xi_1, \xi_2),$$

$$(3.3) \quad \langle \xi_i, \xi_j \rangle = \delta_{ij}, \quad i, j = 1, 2.$$

Since Δ is nonintegrable,

$$(3.4) \quad \xi_3 := [\xi_1, \xi_2] \notin \Delta,$$

and $\mathfrak{g} = \text{span}(\xi_1, \xi_2, \xi_3)$. Then $[\mathfrak{g}, \mathfrak{g}] = \mathbb{R}\xi_3$, and by virtue of (3.1), $\mathbb{R}\xi_3$ is the center of $\mathfrak{g}$:

$$(3.5) \quad [\xi_3, \xi_1] = 0, \quad [\xi_3, \xi_2] = 0.$$

Consequently, for any flat sub-Riemannian structure on G one can choose a basis ξ_1, ξ_2, ξ_3 in $\mathfrak{g}$ with the multiplication rules (3.4), (3.5). Thus any flat sub-Riemannian structures on the Heisenberg group are isomorphic one to another; the more so this is true for flat distributions.

Any basis ξ_1, ξ_2, ξ_3 in the Heisenberg algebra satisfying conditions (3.4) and (3.5) defines a graduation:

$$\mathfrak{g} = \mathfrak{g}^1 \oplus \mathfrak{g}^2, \quad \mathfrak{g}^1 = \text{span}(\xi_1, \xi_2), \quad \mathfrak{g}^2 = \text{span}(\xi_3).$$

The multiplication rules (3.4), (3.5) in the Heisenberg algebra $\mathfrak{g}$ are schematically shown in Figure 1.

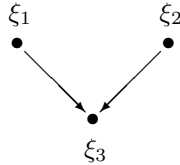


FIGURE 1. The Heisenberg algebra

The Heisenberg group can be represented by 3×3 upper diagonal matrices:

$$G \cong \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\}$$

with the usual matrix multiplication. This linear representation gives rise to another model of the Heisenberg group:

$$G \cong \mathbb{R}_{x,y,z}^3$$

via the mapping

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}_{x,y,z}^3.$$

Multiplication in the Lie group $\mathbb{R}_{x,y,z}^3$ is then given by

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \cdot \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 + x_1 y_2 \end{pmatrix},$$

and the vector fields

$$(3.6) \quad \begin{aligned} \xi_1 &= \frac{\partial}{\partial x}, \\ \xi_2 &= \frac{\partial}{\partial y} + x \frac{\partial}{\partial z}, \\ \xi_3 &= \frac{\partial}{\partial z} \end{aligned}$$

form a basis of the Lie algebra of left-invariant vector fields on $\mathbb{R}_{x,y,z}^3$.

Thus we have a model of the Heisenberg group as $\mathbb{R}_{x,y,z}^3$, and the vector fields ξ_1, ξ_2 in (3.6) give a representation of the flat distribution and the flat sub-Riemannian structure in this model since equalities (3.4) and (3.5) are verified.

Now we compute the Lie algebras of symmetries $\text{Sym}(\Delta)$ and $\text{Sym}(\Delta, \langle \cdot, \cdot \rangle)$ with the help of this model.

3.2. Symmetries of the distribution.

Theorem 1. *The Lie algebra of symmetries of the flat distribution Δ on the Heisenberg group is parametrized by arbitrary smooth functions of three variables.*

For the model in $\mathbb{R}_{x,y,z}^3$ given by (3.2), (3.6),

$$\text{Sym}(\Delta) = \left\{ X = P \frac{\partial}{\partial x} + Q \frac{\partial}{\partial y} + R \frac{\partial}{\partial z} \right\}$$

with

$$\begin{aligned} P &= -f_y - x f_z, \\ Q &= f_x, \\ R &= x f_x - f, \end{aligned}$$

where $f = f(x, y, z)$ is an arbitrary smooth function.

The function f is called the generating function of the symmetry X .

Remark 2. It is well known that locally all contact structures in $\mathbb{R}^3$ (i.e., nonintegrable rank two distributions in $\mathbb{R}^3$) are isomorphic. Thus Theorem 1 describes symmetries of a germ of a contact structure in $\mathbb{R}^3$.

Proof. Take an arbitrary vector field in $\mathbb{R}_{x,y,z}^3$: $X = P \frac{\partial}{\partial x} + Q \frac{\partial}{\partial y} + R \frac{\partial}{\partial z}$, where P , Q , and R are functions of x, y, z . We have

$$(3.7) \quad [\xi_1, X] = P_x \frac{\partial}{\partial x} + Q_x \frac{\partial}{\partial y} + R_x \frac{\partial}{\partial z},$$

$$(3.8) \quad [\xi_2, X] = E_P \frac{\partial}{\partial x} + E_Q \frac{\partial}{\partial y} + (E_R - P) \frac{\partial}{\partial z},$$

where

$$E_P = P_y + xP_z, \quad E_Q = Q_y + xQ_z, \quad E_R = R_y + xR_z.$$

We denote by P_x the partial derivative $\frac{\partial P}{\partial x}$, etc. Thus statement (1) of Proposition 1 reads as follows:

$$\begin{pmatrix} P_x \\ Q_x \\ R_x \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \\ x \end{pmatrix}, \quad \begin{pmatrix} E_P \\ E_Q \\ E_R - P \end{pmatrix} = \gamma \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \delta \begin{pmatrix} 0 \\ 1 \\ x \end{pmatrix}$$

for some real-valued functions $\alpha, \beta, \gamma, \delta$. This system of two vector equations is compatible iff the following system of scalar equations holds:

$$(3.9) \quad R_x = xQ_x,$$

$$(3.10) \quad P = E_R - xE_Q.$$

We integrate the first equation by parts:

$$R = \int xQ_x dx = xQ - \int Q dx$$

and denote $f = \int Q dx$. Then

$$(3.11) \quad Q = f_x,$$

$$(3.12) \quad R = xQ - f = xf_x - f,$$

$$(3.13) \quad P = R_y + xR_z - x(Q_y + xQ_z) = -f_y - xf_z.$$

Thus system (3.9), (3.10) implies system (3.11)–(3.13) for some function $f = f(x, y, z)$. Consequently, if $X \in \text{Sym}(\Delta)$, then system (3.11)–(3.13) holds for some f . Conversely, it is easy to verify that for an arbitrary function f the vector field $X = P \frac{\partial}{\partial x} + Q \frac{\partial}{\partial y} + R \frac{\partial}{\partial z}$ determined by system (3.11)–(3.13) is a symmetry of the distribution Δ .

The correspondence between symmetries X and their generating functions f is one-to-one since $f = xQ - R$. $\square$

3.3. Symmetries of the sub-Riemannian structure.

Theorem 2. *Symmetries of the flat sub-Riemannian structure $(\Delta, \langle \cdot, \cdot \rangle)$ on the Heisenberg group form the four-dimensional Diamond Lie algebra*

$$\text{Sym}(\Delta, \langle \cdot, \cdot \rangle) = \text{span}(X_0, X_1, X_2, X_3)$$

with the multiplication rules

$$(3.14) \quad [X_0, X_1] = -X_2, \quad [X_0, X_2] = X_1, \quad [X_1, X_2] = X_3.$$

For the model in $\mathbb{R}_{x,y,z}^3$ given by (3.2), (3.3), and (3.6), we have

$$(3.15) \quad X_0 = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} + \frac{1}{2}(x^2 - y^2) \frac{\partial}{\partial z},$$

$$(3.16) \quad X_1 = \frac{\partial}{\partial x} + y \frac{\partial}{\partial z},$$

$$(3.17) \quad X_2 = \frac{\partial}{\partial y},$$

$$(3.18) \quad X_3 = -\frac{\partial}{\partial z}.$$

Remarks. (1) Multiplication rules (3.14) in the Lie algebra $\text{Sym}(\Delta, \langle \cdot, \cdot \rangle)$ are schematically represented in Figure 2, which explains the title Diamond for this Lie algebra.

(2) In terms of Theorem 1, the symmetries $X_0, \dots, X_3$ have the following generating functions respectively:

$$f_0 = \frac{x^2 + y^2}{2}, \quad f_1 = -y, \quad f_2 = x, \quad f_3 = 1.$$

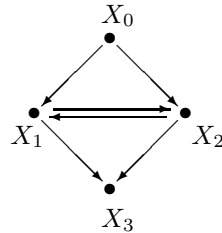


FIGURE 2. $\text{Sym}(\Delta, \langle \cdot, \cdot \rangle)$, the Heisenberg case

The symmetries X_1, X_2, X_3 are just left translations on the Heisenberg group G (compare with Fig. 1), while X_0 is a rotation on G , i.e., a symmetry leaving the identity of G fixed.

Proof. By virtue of the commutation relations (3.7), (3.8), statement (2) of Proposition 1 takes the form

$$\begin{pmatrix} P_x \\ Q_x \\ R_x \end{pmatrix} = \beta \begin{pmatrix} 0 \\ 1 \\ x \end{pmatrix}, \quad \begin{pmatrix} E_P \\ E_Q \\ E_R - P \end{pmatrix} = -\beta \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

for some real-valued function β . This vector system implies the following one:

$$(3.19) \quad R_x = xQ_x,$$

$$(3.20) \quad P = E_R,$$

$$(3.21) \quad P_x = 0,$$

$$(3.22) \quad E_Q = 0,$$

$$(3.23) \quad E_P = -Q_x.$$

As in Theorem 1, we take $f = f(x, y, z) = \int Q dx$. This gives, together with equations (3.19) and (3.20) that

$$(3.24) \quad Q = f_x,$$

$$(3.25) \quad R = xQ - f = xf_x - f,$$

$$(3.26) \quad P = R_y + xR_z = xf_{xy} - f_y + x^2f_{xz} - xf_z.$$

Equations (3.26), (3.21), (3.24), and (3.22) imply that

$$(3.27) \quad \begin{aligned} P_x &= xf_{xxy} + xf_{xz} + x^2f_{xxz} - f_z = 0, \\ E_Q &= f_{xy} + xf_{xz} = 0. \end{aligned}$$

That is why

$$P_x - x(E_Q)_x = -f_z = 0;$$

thus,

$$f = f(x, y).$$

Then equation (3.27) gives $f_{xy} = 0$, which means that

$$f = a(x) + b(y).$$

We obtain from (3.26) that $P = -b_y$; hence $E_P = P_y + xP_z = -b_{yy}$. On the other hand, equation (3.24) gives $Q = a_x$. Now (3.23) takes the form $-b_{yy} = -a_{xx}$. But the right-hand side of this equality depends on y , whereas the left-hand one depends on x , which means that they both are constant. We denote this constant by $-c$ and obtain

$$\begin{aligned} b &= \frac{c}{2}y^2 + dy + e, \\ a &= \frac{c}{2}x^2 + gx \end{aligned}$$

for some $c, d, e, g \in \mathbb{R}$. Hence

$$f = \frac{c}{2}(x^2 + y^2) + dy + gx + e$$

and

$$(3.28) \quad X = (-cy - d)\frac{\partial}{\partial x} + (cx + g)\frac{\partial}{\partial y} + \left(\frac{c}{2}(x^2 - y^2) - dy - e\right)\frac{\partial}{\partial z}.$$

To summarize the above computations, if a vector field X is a symmetry of our sub-Riemannian structure, then it has the form (3.28). The converse statement is verified immediately. That is why

$$\begin{aligned} \text{Sym}(\Delta, \langle \cdot, \cdot \rangle) = \\ \left\{ (-cy - d)\frac{\partial}{\partial x} + (cx + g)\frac{\partial}{\partial y} + \left(\frac{c}{2}(x^2 - y^2) - dy - e\right)\frac{\partial}{\partial z} \mid c, d, e, g \in \mathbb{R} \right\}. \end{aligned}$$

Now we compute a basis of the 4-dimensional Lie algebra $\text{Sym}(\Delta, \langle \cdot, \cdot \rangle)$.

$$\begin{aligned} c = 1, d = e = g = 0 &\Rightarrow X_0 = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} + \frac{1}{2}(x^2 - y^2) \frac{\partial}{\partial z}, \\ c = 0, d = -1, e = g = 0 &\Rightarrow X_1 = \frac{\partial}{\partial x} + y \frac{\partial}{\partial z}, \\ c = d = e = 0, g = 1 &\Rightarrow X_2 = \frac{\partial}{\partial y}, \\ c = d = 0, e = 1, g = 0 &\Rightarrow X_3 = -\frac{\partial}{\partial z}. \end{aligned}$$

Nonzero brackets between the basis vectors are exactly the commutation relations of the Diamond Lie algebra, see (3.14). $\square$

4. THE ENGEL CASE

4.1. The Engel algebra and Engel group. Let $\mathfrak{g}$ be the Engel algebra, i.e., the (unique) four-dimensional 3-step nilpotent Lie algebra, and let G be the Engel group, i.e., the corresponding connected simply connected Lie group. There is a basis $\xi_1, \xi_2, \xi_3, \xi_4$ in $\mathfrak{g}$ with the only nonzero brackets

$$[\xi_1, \xi_2] = \xi_3, \quad [\xi_2, \xi_3] = \xi_4.$$

In this basis, multiplication in the Engel algebra is schematically represented by the diagram in Figure 3.

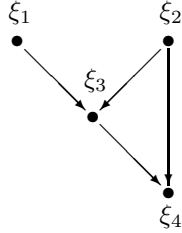


FIGURE 3. The Engel algebra

The Engel algebra is graded:

$$\mathfrak{g} = \mathfrak{g}^1 \oplus \mathfrak{g}^2 \oplus \mathfrak{g}^3,$$

where

$$\mathfrak{g}^1 = \text{span}(\xi_1, \xi_2), \quad \mathfrak{g}^2 = \text{span}(\xi_3), \quad \mathfrak{g}^3 = \text{span}(\xi_4).$$

4.2. The flat distribution and sub-Riemannian structure. As was shown in [7], all flat distributions on the Engel group are isomorphic. Now we prove that, moreover, all flat sub-Riemannian structures on the Engel group are also isomorphic.

Let $(\Delta, \langle \cdot, \cdot \rangle)$ be a flat sub-Riemannian structure on the Engel group G corresponding to a graduation of $\mathfrak{g}$:

$$\begin{aligned}\mathfrak{g} &= \mathfrak{g}^1 \oplus \mathfrak{g}^2 \oplus \mathfrak{g}^3, \\ \Delta &= \mathfrak{g}^1, \quad \dim \mathfrak{g}^1 = 2, \\ \mathfrak{g}^2 &= [\mathfrak{g}^1, \mathfrak{g}^1], \quad \dim \mathfrak{g}^2 = 1, \\ \mathfrak{g}^3 &= [\mathfrak{g}^1, \mathfrak{g}^2], \quad \dim \mathfrak{g}^3 = 1\end{aligned}$$

(certainly, these homogeneous components $\mathfrak{g}^i$ should not be the same as in the previous subsection, but their number and dimensions are obviously the same).

Choose any nonzero vector

$$\xi_3 \in \mathfrak{g}^2.$$

The operator

$$\text{ad } \xi_3 : \mathfrak{g}^1 \rightarrow \mathfrak{g}^3$$

has one-dimensional image and thus one-dimensional kernel. We can choose an orthonormal frame in $\mathfrak{g}^1$ so that

$$\begin{aligned}\mathfrak{g}^1 &= \text{span}(\xi_1, \xi_2), \\ \langle \xi_i, \xi_j \rangle &= \delta_{ij}, \quad i, j = 1, 2, \\ \ker(\text{ad } \xi_3)|_{\mathfrak{g}^1} &= \text{span}(\xi_1).\end{aligned}\tag{4.1}$$

Moreover,

$$[\xi_1, \xi_2] = k\xi_3, \quad k \in \mathbb{R} \setminus \{0\}.$$

Now we denote $k\xi_3$ as ξ_3 and obtain

$$[\xi_1, \xi_2] = \xi_3.\tag{4.2}$$

Finally, the vector

$$\xi_4 = [\xi_2, \xi_3]\tag{4.3}$$

spans the homogeneous component $\mathfrak{g}^3$. Equality (4.1) and the inclusion $\xi_4 \in \mathfrak{g}^3$ mean that all Lie brackets between the vector fields ξ_1, ξ_2, ξ_3 , and ξ_4 are zero except those given by (4.2) and (4.3). Consequently, any flat sub-Riemannian structure $(\Delta, \langle \cdot, \cdot \rangle)$ on the Engel group possesses an orthonormal frame with the multiplication rules (4.2) and (4.3). (In the sequel we call such a frame a *standard* left-invariant frame on the Engel group.) This proves the uniqueness of flat sub-Riemannian structures up to an isomorphism. The uniqueness of flat distributions obviously follows.

That is why any particular flat distribution and sub-Riemannian structure can be used to compute the Lie algebras of symmetries $\text{Sym}(\Delta)$ and $\text{Sym}(\Delta, \langle \cdot, \cdot \rangle)$.

4.3. The model in $\mathbb{R}^4$. The four-dimensional space $\mathbb{R}_{x,y,z,u}^4$ is the Engel group with the multiplication rule

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \\ u_1 \end{pmatrix} \cdot \begin{pmatrix} x_2 \\ y_2 \\ z_2 \\ u_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 + x_1 y_2 \\ u_1 + u_2 + y_1 z_2 + x_1 y_1 y_2 + x_1 y_2^2 / 2 \end{pmatrix}.$$

Then the vector fields

$$(4.4) \quad \xi_1 = \frac{\partial}{\partial x},$$

$$(4.5) \quad \begin{aligned} \xi_2 &= \frac{\partial}{\partial y} + x \frac{\partial}{\partial z} + xy \frac{\partial}{\partial u}, \\ \xi_3 &= [\xi_1, \xi_2] = \frac{\partial}{\partial z} + y \frac{\partial}{\partial u}, \\ \xi_4 &= [\xi_2, \xi_3] = \frac{\partial}{\partial u} \end{aligned}$$

form a standard left-invariant frame on $\mathbb{R}_{x,y,z,u}^4$.

Thus we have the following model of the flat distribution Δ and sub-Riemannian structure $(\Delta, \langle \cdot, \cdot \rangle)$ on the Engel group in $\mathbb{R}_{x,y,z,u}^4$:

$$(4.6) \quad \Delta = \text{span}(\xi_1, \xi_2),$$

$$(4.7) \quad \langle \xi_i, \xi_j \rangle = \delta_{ij}, \quad i, j = 1, 2.$$

Now we compute the symmetries $\text{Sym}(\Delta)$ and $\text{Sym}(\Delta, \langle \cdot, \cdot \rangle)$ in this model.

4.3.1. Symmetries of the distribution.

Theorem 3. *The Lie algebra of symmetries of the flat distribution Δ on the Engel group is parametrized by functions of 4 variables constant along the canonical vector field.*

For the model in $\mathbb{R}_{x,y,z,u}^4$ given by (4.4)–(4.6), we have

$$\text{Sym}(\Delta) = \left\{ X = S \frac{\partial}{\partial x} + P \frac{\partial}{\partial y} + Q \frac{\partial}{\partial z} + R \frac{\partial}{\partial u} \right\},$$

where

$$(4.8) \quad S = f_{yy} + 2xf_{yz} + 2xyf_{yu} + xf_u + x^2f_{zz} + 2x^2yf_{zu} + x^2y^2f_{uu},$$

$$(4.9) \quad P = -f_z - yf_u,$$

$$(4.10) \quad Q = f_y,$$

$$(4.11) \quad R = yf_y - f,$$

and

$$f = f(y, z, u)$$

is an arbitrary smooth function of the variables y, z, u .

Remark 3. It is known that locally all Engel structures in $\mathbb{R}^4$ (i.e., maximal growth rank two distributions in $\mathbb{R}^4$) are isomorphic. Thus Theorem 3 describes symmetries of a germ of an Engel structure in $\mathbb{R}^4$. This question will be continued in Subsection 4.4.

Proof. Take an arbitrary vector field $X = S \frac{\partial}{\partial x} + P \frac{\partial}{\partial y} + Q \frac{\partial}{\partial z} + R \frac{\partial}{\partial u} \in \text{Vec}(\mathbb{R}^4)$. In view of the equalities

$$\begin{aligned} [\xi_1, X] &= S_x \frac{\partial}{\partial x} + P_x \frac{\partial}{\partial y} + Q_x \frac{\partial}{\partial z} + R_x \frac{\partial}{\partial u}, \\ [\xi_2, X] &= E_S \frac{\partial}{\partial x} + E_P \frac{\partial}{\partial y} + (E_Q - S) \frac{\partial}{\partial z} + (E_R - yS - xP) \frac{\partial}{\partial u}, \end{aligned}$$

where

$$\begin{aligned} E_S &= \xi_2 S = S_y + xS_z + xyS_u, & E_P &= \xi_2 P = P_y + xP_z + xyP_u, \\ E_Q &= \xi_2 Q = Q_y + xQ_z + xyQ_u, & E_R &= \xi_2 R = R_y + xR_z + xyR_u, \end{aligned}$$

and by virtue of Proposition 1, a vector field X is a symmetry of the distribution Δ iff

$$(4.12) \quad \begin{pmatrix} S_x \\ P_x \\ Q_x \\ R_x \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \\ x \\ xy \end{pmatrix},$$

$$(4.13) \quad \begin{pmatrix} E_S \\ E_P \\ E_Q - S \\ E_R - yS - xP \end{pmatrix} = \gamma \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \delta \begin{pmatrix} 0 \\ 1 \\ x \\ xy \end{pmatrix}$$

for some smooth real-valued functions $\alpha, \beta, \gamma, \delta$. These equations for $\alpha, \beta, \gamma, \delta$ are solvable iff the following equalities hold:

$$\begin{aligned} Q_x &= xP_x, \\ R_x &= xyP_x, \\ E_Q - S &= E_P x, \\ E_R - yS - xP &= E_P xy, \end{aligned}$$

which are equivalent to

$$(4.14) \quad Q_x = xP_x,$$

$$(4.15) \quad R_x = yQ_x,$$

$$(4.16) \quad S = E_Q - xE_P,$$

$$(4.17) \quad E_Q y = E_R - xP.$$

Equality (4.15) gives

$$yQ_x = R_x \Leftrightarrow \int yQ_x dx = \int R_x dx \Leftrightarrow yQ = R + f,$$

where

$$f = f(y, z, u)$$

is some smooth function of the variables y, z, u . Thus

$$(4.18) \quad R = yQ - f.$$

Then equality (4.17) gives

$$(4.19) \quad P = \frac{1}{x}(E_R - yE_Q) = \frac{1}{x}(Q - (f_y + xf_z + xyf_u)).$$

Finally, equality (4.14) takes the form

$$Q_x = \frac{1}{x}(Q_x x - Q + f_y),$$

i.e.,

$$(4.20) \quad Q = f_y.$$

This gives, in view of (4.18),

$$(4.21) \quad R = yf_y - f.$$

Now we obtain from (4.19)

$$(4.22) \quad P = -f_z - yf_u,$$

and from (4.16)

$$(4.23) \quad S = f_{yy} + 2xf_{yz} + 2xyf_{yu} + xf_u + x^2f_{zz} + 2x^2yf_{zu} + x^2y^2f_{uu}.$$

The above computations show that if a vector field $X = S\frac{\partial}{\partial x} + P\frac{\partial}{\partial y} + Q\frac{\partial}{\partial z} + R\frac{\partial}{\partial u}$ is a symmetry of the distribution Δ , then its components S, P, Q, R satisfy equalities (4.23), (4.22), (4.20), (4.21) for some function $f = f(y, z, u)$.

Direct computation shows that given an arbitrary smooth function $f = f(y, z, u)$, any vector field $X = S\frac{\partial}{\partial x} + P\frac{\partial}{\partial y} + Q\frac{\partial}{\partial z} + R\frac{\partial}{\partial u}$ with the components S, P, Q, R determined from equalities (4.23), (4.22), (4.20), (4.21) belongs to $\text{Sym}(\Delta)$.

The correspondence between symmetries X and generating functions f is one-to-one since $f = yQ - R$. $\square$

4.3.2. Symmetries of the sub-Riemannian structure.

Theorem 4. *The Lie algebra of symmetries of the flat sub-Riemannian structure $(\Delta, \langle \cdot, \cdot \rangle)$ on the Engel group is the Engel algebra.*

For the model in $\mathbb{R}^4_{x,y,z,u}$ defined by (4.4)–(4.7) we have

$$\text{Sym}(\Delta, \langle \cdot, \cdot \rangle) = \text{span}(X_1, X_2, X_3, X_4),$$

where

$$\begin{aligned} X_1 &= \frac{\partial}{\partial x} + y\frac{\partial}{\partial z} + \frac{1}{2}y^2\frac{\partial}{\partial u}, \\ X_2 &= \frac{\partial}{\partial y} + z\frac{\partial}{\partial u}, \\ X_3 &= -\frac{\partial}{\partial z}, \\ X_4 &= \frac{\partial}{\partial u}. \end{aligned}$$

Remark 4. Nonzero Lie brackets in the Lie algebra $\text{Sym}(\Delta, \langle \cdot, \cdot \rangle)$ are described by the scheme in Figure 4; compare with the scheme for the Engel algebra in Figure 3.

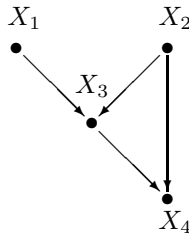


FIGURE 4. $\text{Sym}(\Delta, \langle \cdot, \cdot \rangle)$, the Engel case

Proof. If a vector field $X = S\frac{\partial}{\partial x} + P\frac{\partial}{\partial y} + Q\frac{\partial}{\partial z} + R\frac{\partial}{\partial u} \in \text{Vec}(\mathbb{R}^4)$ is a symmetry of the sub-Riemannian structure $(\Delta, \langle \cdot, \cdot \rangle)$, then equalities (4.12) and (4.13) should hold with

$$\alpha = \delta = 0, \quad \beta = -\gamma.$$

That is why S, P, Q, R satisfy both the old equations (4.14)–(4.17), which mean that X is a symmetry of the distribution Δ , and the following additional equations, which mean that X preserves the inner product $\langle \cdot, \cdot \rangle$ as well:

$$(4.24) \quad S_x = 0,$$

$$(4.25) \quad E_S = -P_x,$$

$$(4.26) \quad E_P = 0.$$

That is why, by Theorem 3, for the components S, P, Q, R , equations (4.8)–(4.11) hold.

Equations (4.8) and (4.24) give

$$S_x = 2f_{yz} + 2yf_{yu} + f_u + 2xf_{zz} + 4xyf_{zu} + 2xy^2f_{uu} = 0.$$

But f does not depend on x ; thus we decompose the previous equality in powers of x :

$$(4.27) \quad 2f_{yz} + f_u + 2yf_{yu} = 0,$$

$$(4.28) \quad 2f_{zz} + 4yf_{zu} + 2y^2f_{uu} = 0.$$

Analogously, equalities (4.10) and (4.26) lead to

$$f_{zy} + f_u + yf_{yu} + xf_{zz} + 2xyf_{zu} + xy^2f_{uu} = 0,$$

which decomposes into powers of x :

$$(4.29) \quad f_{zy} + f_u + yf_{yu} = 0,$$

$$(4.30) \quad f_{zz} + 2yf_{zu} + y^2f_{uu} = 0.$$

We subtract equation (4.27) from the doubled equation (4.29) and obtain

$$f_u = 0,$$

which means

$$f = f(y, z).$$

Now equations (4.29) and (4.30) read

$$(4.31) \quad f_{yz} = 0,$$

$$(4.32) \quad f_{zz} = 0.$$

Then condition (4.31) is equivalent to

$$f = a(y) + b(z),$$

and equality (4.32) gives

$$b(z) = bz + c, \quad b, c \in \mathbb{R}.$$

Consequently,

$$(4.33) \quad f = a(y) + bz + c.$$

Finally, we obtain from (4.8) and (4.33) that

$$S = a''(y),$$

and from (4.9) that

$$P = -b.$$

Then the last yet unused additional equation (4.25) yields

$$a'''(y) = 0;$$

thus,

$$a(y) = ay^2 + dy + \text{const}, \quad a, d \in \mathbb{R}.$$

That is why

$$f = ay^2 + dy + bz + c, \quad a, b, c, d \in \mathbb{R}.$$

Now we recover the components S, P, Q, R from (4.8)–(4.11) and obtain that any vector field $X \in \text{Sym}(\Delta, \langle \cdot, \cdot \rangle)$ must have the form

$$(4.34) \quad X = 2a \frac{\partial}{\partial x} - b \frac{\partial}{\partial y} + (2ay + d) \frac{\partial}{\partial z} + (ay^2 - bz - c) \frac{\partial}{\partial u}, \quad a, b, c, d \in \mathbb{R}.$$

Direct computation verifies that for any $a, b, c, d \in \mathbb{R}$ the vector field X given by (4.34) is a symmetry of the sub-Riemannian structure $(\Delta, \langle \cdot, \cdot \rangle)$. So the Lie algebra $\text{Sym}(\Delta, \langle \cdot, \cdot \rangle)$ is 4-dimensional. We compute its basis:

$$\begin{aligned} a = \frac{1}{2}, \quad b = c = d = 0 &\Rightarrow X_1 = \frac{\partial}{\partial x} + y \frac{\partial}{\partial z} + \frac{1}{2} y^2 \frac{\partial}{\partial u}, \\ a = 0, \quad b = -1, \quad c = d = 0 &\Rightarrow X_2 = \frac{\partial}{\partial y} + z \frac{\partial}{\partial u}, \\ a = b = c = 0, \quad d = -1 &\Rightarrow X_3 = -\frac{\partial}{\partial z}, \\ a = b = 0, \quad c = -1, \quad d = 0 &\Rightarrow X_4 = \frac{\partial}{\partial u}. \end{aligned}$$

Nonzero brackets between the basis vectors are:

$$[X_1, X_2] = X_3, \quad [X_2, X_3] = X_4.$$

Consequently, $\text{Sym}(\Delta, \langle \cdot, \cdot \rangle) = \text{span}(X_1, X_2, X_3, X_4)$ is the Engel algebra. $\square$

4.4. Engel structure and a transverse contact structure. An Engel structure on a four-dimensional manifold M_4 (see e.g. [7]) is a rank two maximal growth distribution Δ on M_4 , i.e., a rank two distribution Δ with the growth vector $(2, 3, 4)$.

One of the vector fields admissible for an Engel distribution Δ , namely ξ_1 , satisfies the property

$$(\text{ad } \xi_1) \Delta^2 \subset \Delta^2.$$

This property determines the vector field ξ_1 uniquely up to a nonvanishing factor. Such a vector field is called a *canonical vector field* of the distribution Δ .

Given an Engel distribution Δ on a four-dimensional manifold M_4 , its canonical vector field ξ_1 , and a three-dimensional submanifold $N_3 \subset M_4$ transversal to ξ_1 , the distribution

$$D = \Delta^2 \cap TN_3$$

defines a contact structure on N_3 (see [7]) called a *transverse contact structure*.

Locally, all Engel structures are isomorphic (see [5], [7]); in particular, a germ of any Engel structure can be represented by the model

$$\Delta = \text{span}(\xi_1, \xi_2), \quad \xi_1, \xi_2 \in \text{Vec}(\mathbb{R}_{x,y,z,u}^4)$$

considered in Subsection 4.3. This implies that the Lie algebras of symmetries of the flat Engel distribution computed in Theorem 3 are Lie algebras of symmetries of a germ of an arbitrary Engel distribution.

On the other hand, all contact structures are also locally isomorphic (the Darboux theorem). In particular, any contact structure on a three-dimensional manifold is locally isomorphic to the flat rank two distribution on the Heisenberg group (see Subsection 3.1), and the Lie algebra of symmetries computed in Theorem 1 is, in fact, the Lie algebra of symmetries of a germ of a contact structure on a three-dimensional manifold.

That is why, comparing Theorems 1 and 3, we arrive at the following proposition.

Theorem 5. *Let Δ be a germ of an Engel distribution on a four-dimensional manifold M_4 with a canonical vector field ξ_1 , and let D be a germ of a transverse contact structure on a three-dimensional submanifold $N_3 \subset M_4$ transversal to ξ_1 . Then there is a one-to-one correspondence between:*

- 1) *symmetries of Δ ;*
- 2) *symmetries of D ;*
- 3) *functions $f : M_4 \rightarrow \mathbb{R}$ constant along the canonical vector field ξ_1 .*

5. THE CARTAN CASE

Rank two distributions in the five-dimensional space were studied by E. Cartan [6], and this gave the title of the case we consider in this section.

5.1. The Lie algebra and Lie group. Let $\mathfrak{g}$ be the five-dimensional nilpotent three-step Lie algebra with multiplication rules in some basis

$$\mathfrak{g} = \text{span}(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5)$$

as follows:

$$[\xi_1, \xi_2] = \xi_3, \quad [\xi_1, \xi_3] = \xi_4, \quad [\xi_2, \xi_3] = \xi_5$$

(all the remaining brackets are equal to zero). We call such a frame $\xi_1, \dots, \xi_5$ a *standard* left-invariant frame on $\mathfrak{g}$. Multiplication rules in a standard frame of $\mathfrak{g}$ are represented by the scheme in Figure 5.

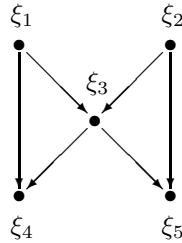


FIGURE 5. Lie algebra $\mathfrak{g}$, the Cartan case

The Lie algebra $\mathfrak{g}$ is graded:

$$\mathfrak{g} = \mathfrak{g}^1 \oplus \mathfrak{g}^2 \oplus \mathfrak{g}^3,$$

where

$$\mathfrak{g}^1 = \text{span}(\xi_1, \xi_2), \quad \mathfrak{g}^2 = \text{span}(\xi_3), \quad \mathfrak{g}^3 = \text{span}(\xi_4, \xi_5).$$

Denote by G the simply connected Lie group corresponding to $\mathfrak{g}$.

5.2. The flat distribution and sub-Riemannian structure. We assert that any flat distribution or sub-Riemannian structure on the Lie group G is isomorphic to the following one defined via a standard frame in $\mathfrak{g}$:

$$(5.1) \quad \Delta = \text{span}(\xi_1, \xi_2), \quad \langle \xi_i, \xi_j \rangle = \delta_{ij}, \quad i, j = 1, 2.$$

As before, we prove the isomorphism for sub-Riemannian structures, and the isomorphism for distributions will follow. Take an arbitrary flat sub-Riemannian structure $(\Delta, \langle \cdot, \cdot \rangle)$ on the Lie group G corresponding to a graduation of $\mathfrak{g}$:

$$\begin{aligned} \mathfrak{g} &= \mathfrak{g}^1 \oplus \mathfrak{g}^2 \oplus \mathfrak{g}^3, \\ \Delta &= \mathfrak{g}^1, \quad \dim \mathfrak{g}^1 = 2, \\ \mathfrak{g}^2 &= [\mathfrak{g}^1, \mathfrak{g}^1], \quad \dim \mathfrak{g}^2 = 1, \\ \mathfrak{g}^3 &= [\mathfrak{g}^1, \mathfrak{g}^2], \quad \dim \mathfrak{g}^3 = 2 \end{aligned}$$

(as in the Engel case, these homogeneous components $\mathfrak{g}^i$ should not be the same as in the previous subsection, but their number and dimensions are obviously the same).

Choose any orthonormal basis as in (5.1). Then the vector

$$\xi_3 = [\xi_1, \xi_2]$$

spans $\mathfrak{g}^2$, and the vectors

$$\xi_4 = [\xi_1, \xi_3], \quad \xi_5 = [\xi_2, \xi_3]$$

span the homogeneous component $\mathfrak{g}^3$. Thus the orthonormal frame ξ_1, ξ_2 of the sub-Riemannian structure $(\Delta, \langle \cdot, \cdot \rangle)$ generates a standard frame in $\mathfrak{g}$. This proves the uniqueness of flat sub-Riemannian structures on the Lie group G up to an isomorphism; the flat distributions are isomorphic so much the more.

5.3. The Cartan model. In this subsection we describe the local model of the flat distribution on the Lie group G due to E. Cartan (this construction was kindly communicated to us by A. A. Agrachev).

Let $\mathfrak{g} = \mathfrak{g}_2$ be the (unique) noncompact real form of the complex simple 14-dimensional Lie algebra $\mathfrak{g}_2^{\mathbb{C}}$, and let G be the corresponding connected simply connected Lie group. We consider $\mathfrak{g}$ as the Lie algebra of left-invariant vector fields on G and choose a basis

$$\mathfrak{g} = \text{span}(Z_1, \dots, Z_{14})$$

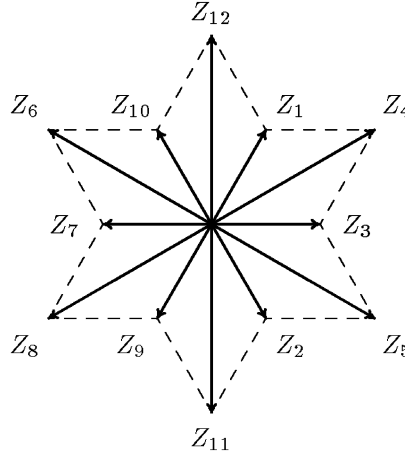
so that Z_{13} and Z_{14} span a (two-dimensional) Cartan subalgebra of $\mathfrak{g}$ and $Z_1, \dots, Z_{12}$ correspond to root vectors; see Figure 6. (Compare with the description of the linear representation of $\mathfrak{g}_2$ in the Appendix and Figure 8.)

The vector fields Z_1, Z_2 generate the left-invariant 2-distribution

$$D = \text{span}(Z_1, Z_2)$$

on G and the 5-dimensional nilpotent Lie algebra

$$\mathfrak{n} = \text{Lie}(Z_1, Z_2) = \text{span}(Z_1, \dots, Z_5),$$

FIGURE 6. Root vectors of $\mathfrak{g}$, the Cartan case

which is obvious from the scheme of root vectors in Figure 6. The same figure shows that the vector subspace of $\mathfrak{g}$ transversal to $\mathfrak{n}$, namely

$$\mathfrak{h} = \text{span}(Z_6, \dots, Z_{14}),$$

is in fact a subalgebra having the property

$$(5.2) \quad \text{ad } \mathfrak{h}(D) \subset D + \mathfrak{h}.$$

That is, the subalgebra $\mathfrak{h}$ preserves the distribution D modulo $\mathfrak{h}$ itself; consequently, we can factorize by $\mathfrak{h}$.

Let H be the local connected subgroup of G corresponding to the Lie subalgebra $\mathfrak{h}$, and

$$M = G/H = \{ xH \mid x \in G \}$$

the left coset space, a smooth 5-dimensional manifold. Take the corresponding projection and its differential

$$\pi : G \rightarrow G/H = M, \quad \pi : x \mapsto xH, \quad \pi_* : \mathfrak{h} \rightarrow \mathfrak{g}/\mathfrak{h}$$

and define

$$\Delta = \pi_*(D).$$

By virtue of (5.2), Δ is a correctly defined 2-distribution on M .

We choose a vector field basis in Δ as follows. Let N be the local subgroup of G corresponding to the nilpotent subalgebra $\mathfrak{n}$. Since

$$\mathfrak{n} \cap \mathfrak{h} = \{0\},$$

then the restriction

$$\pi|_N : N \rightarrow M$$

is a diffeomorphism. Denote the inverse diffeomorphism by

$$\tau : M \rightarrow N, \quad \tau = (\pi|_N)^{-1}$$

and define the vector fields on M by

$$\xi_i(q) = \pi_* Z_i(x), \quad x = \tau(q), \quad i = 1, 2.$$

First,

$$\Delta_q = \text{span}(\xi_1(q), \xi_2(q)), \quad q \in M.$$

Second, the vector fields ξ_i , $i = 1, 2$, are π -related to the vector fields Z_i , $i = 1, 2$, respectively. Consequently,

$$\text{Lie}(\xi_1, \xi_2) \cong \text{Lie}(Z_1, Z_2) = \mathfrak{n}.$$

To summarize, $\Delta \subset TM$ is a 2-distribution on the 5-dimensional manifold M , and admissible vector fields of D form the 5-dimensional nilpotent Lie algebra $\mathfrak{n}$. That is why the distribution Δ is a (local) model for a flat (2,5)-distribution. We call it the *Cartan model*.

Now we compute some symmetries of the distribution Δ with the help of the Cartan model.

The left-invariant distribution D is preserved by all left translations on the Lie group G . The flow of a *right*-invariant vector field on G is realized by *left* translations on G ; that is why all right-invariant vector fields on G are infinitesimal symmetries of D :

$$\mathfrak{g}_r \subset \text{Sym}(D)$$

(we denote by $\mathfrak{g}_r$ the Lie algebra of right-invariant vector fields on G).

Now we project these symmetries to M . The action of the group G by left translations is naturally projected from G to its homogeneous space $M = G/H$; hence, right-invariant vector fields on G are correctly projected to the vector fields

$$\pi_*(\mathfrak{g}_r) \subset \text{Vec}(M).$$

The left translations of G on M preserve the distribution Δ ; thus,

$$\pi_*(\mathfrak{g}_r) \subset \text{Sym}(\Delta).$$

In order to show that the projection π_* does not send any symmetry from $\mathfrak{g}_r$ to zero, suppose the contrary:

$$\exists v \in \mathfrak{g}_r : v(x) \in \ker \pi_*|_x \quad \forall x \in G.$$

We apply this inclusion at the identity $e \in G$ and see that

$$v \in \mathfrak{h}_r$$

(we denote by $\mathfrak{h}_r$ the Lie algebra of all right-invariant vector fields on G that are tangent at the identity e to the subgroup H). On the other hand,

$$\ker \pi_*|_x = \mathfrak{h}(x)$$

(in the right-hand side stands the vector space obtained by the values of the left-invariant vector fields from $\mathfrak{h}$ at the point $x \in G$). Consequently,

$$v \in \mathfrak{h}_r \cap \mathfrak{h}.$$

But

$$\mathfrak{h}_r \cap \mathfrak{h} = \{0\}$$

since the Lie algebra $\mathfrak{g}$ is simple. This means that $v = 0$. Thus,

$$\dim \pi_*(\mathfrak{g}_r) = 14$$

and

$$(5.3) \quad \pi_*(\mathfrak{g}_r) \cong \mathfrak{g}_r \cong \mathfrak{g} \subset \text{Sym}(\Delta).$$

That is, the Cartan model yields that locally a flat (2,5)-distribution has a 14-dimensional algebra of symmetries isomorphic to the Lie algebra $\mathfrak{g}_2$. In Subsubsection 5.4.1 we show that inclusion (5.3) is in fact equality.

5.4. The model in $\mathbb{R}^5$. The five-dimensional space $\mathbb{R}_{x,y,z,u,v}^5$ endowed with the multiplication rule

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \\ u_1 \\ v_1 \end{pmatrix} \cdot \begin{pmatrix} x_2 \\ y_2 \\ z_2 \\ u_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 + x_1 y_2 \\ u_1 + u_2 + z_1 y_2 + x_1 y_2^2/2 \\ v_1 + v_2 + 2x_1 z_2 + x_1^2 y_2 \end{pmatrix}$$

becomes the five-dimensional nilpotent Lie group G described in Subsection 5.1 with the standard left-invariant frame

$$(5.4) \quad \xi_1 = \frac{\partial}{\partial x},$$

$$(5.5) \quad \xi_2 = \frac{\partial}{\partial y} + x \frac{\partial}{\partial z} + z \frac{\partial}{\partial u} + x^2 \frac{\partial}{\partial v},$$

$$\xi_3 = [\xi_1, \xi_2] = \frac{\partial}{\partial z} + 2x \frac{\partial}{\partial v},$$

$$\xi_4 = [\xi_1, \xi_3] = 2 \frac{\partial}{\partial v},$$

$$\xi_5 = [\xi_2, \xi_3] = -\frac{\partial}{\partial u}.$$

Thus we can use the corresponding model of the flat distribution and sub-Riemannian structure on the group G :

$$(5.6) \quad \Delta = \text{span}(\xi_1, \xi_2),$$

$$(5.7) \quad \langle \xi_i, \xi_j \rangle = \delta_{ij}, \quad i, j = 1, 2.$$

Now we compute the symmetries $\text{Sym}(\Delta)$ and $\text{Sym}(\Delta, \langle \cdot, \cdot \rangle)$ with the help of this model.

5.4.1. Symmetries of the distribution.

Theorem 6. *The Lie algebra of symmetries of the flat distribution Δ on the five-dimensional nilpotent Lie group G described in Subsection 5.1 is the 14-dimensional Lie algebra $\mathfrak{g}_2$, that is, the (unique) noncompact real form of the complex exceptional simple Lie algebra $\mathfrak{g}_2^{\mathbb{C}}$.*

For the model of Δ in $\mathbb{R}^5$ defined by (5.4)–(5.6), we have

$$\text{Sym}(\Delta) = \text{span}(Y_1, \dots, Y_{14}),$$

where

$$\begin{aligned}
Y_1 &= \frac{1}{36} \left(2 \frac{\partial}{\partial x} + 2y \frac{\partial}{\partial z} + y^2 \frac{\partial}{\partial u} + 4z \frac{\partial}{\partial v} \right), \\
Y_2 &= -3 \frac{\partial}{\partial y}, \\
Y_3 &= \frac{1}{12} \left(\frac{\partial}{\partial z} + y \frac{\partial}{\partial u} \right), \\
Y_4 &= -\frac{1}{324} \frac{\partial}{\partial v}, \\
Y_5 &= \frac{1}{12} \frac{\partial}{\partial u}, \\
Y_6 &= \frac{1}{3} \left((6x^2y^2 - 12xyz + 24z^2 - 18yv) \frac{\partial}{\partial x} + (6xy^3 - 24zy^2 + 36yu) \frac{\partial}{\partial y} \right. \\
&\quad \left. + (3x^2y^3 - 12yz^2 - 9y^2v + 36zu) \frac{\partial}{\partial z} \right. \\
&\quad \left. + (6xy^3z - 12y^2z^2 - 3y^3v + 36u^2) \frac{\partial}{\partial u} \right. \\
&\quad \left. + (2x^3y^3 - 36yzv + 16z^3 + 36uv) \frac{\partial}{\partial v} \right), \\
Y_7 &= 3 \left((-4x^2y + 4xz + 6v) \frac{\partial}{\partial x} + (-6xy^2 + 16yz - 12u) \frac{\partial}{\partial y} \right. \\
&\quad \left. + (-3x^2y^2 + 4z^2 + 6vy) \frac{\partial}{\partial z} + (-6xy^2z + 8yz^2 + 3y^2v) \frac{\partial}{\partial u} \right. \\
&\quad \left. + (-2x^3y^2 + 12zv) \frac{\partial}{\partial v} \right), \\
Y_8 &= 36 \left((6x^2z - 9xv) \frac{\partial}{\partial x} + (18xu - 12z^2) \frac{\partial}{\partial y} + (9x^2u - 9zv) \frac{\partial}{\partial z} \right. \\
&\quad \left. + (18xzu - 8z^3 - 9uv) \frac{\partial}{\partial u} + (6x^3u - 9v^2) \frac{\partial}{\partial v} \right), \\
Y_9 &= 9 \left(2x^2 \frac{\partial}{\partial x} + (6xy - 8z) \frac{\partial}{\partial y} + (3x^2y - 3v) \frac{\partial}{\partial z} \right. \\
&\quad \left. + (6xyz - 4z^2 - 3yv) \frac{\partial}{\partial u} + 2x^3y \frac{\partial}{\partial v} \right), \\
Y_{10} &= \frac{1}{3} \left((xy - 4z) \frac{\partial}{\partial x} - y^2 \frac{\partial}{\partial y} - (zy + 3u) \frac{\partial}{\partial z} - 3yu \frac{\partial}{\partial u} - 4z^2 \frac{\partial}{\partial v} \right), \\
Y_{11} &= -9 \left(6x \frac{\partial}{\partial y} + 3x^2 \frac{\partial}{\partial z} + (6xz - 3v) \frac{\partial}{\partial u} + 2x^3 \frac{\partial}{\partial v} \right), \\
Y_{12} &= \frac{1}{324} \left(6y \frac{\partial}{\partial x} + 3y^2 \frac{\partial}{\partial z} + y^3 \frac{\partial}{\partial u} + (12yz - 12u) \frac{\partial}{\partial v} \right), \\
Y_{13} &= -x \frac{\partial}{\partial x} - z \frac{\partial}{\partial z} - u \frac{\partial}{\partial u} - 2v \frac{\partial}{\partial v}, \\
Y_{14} &= -x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + u \frac{\partial}{\partial u} - v \frac{\partial}{\partial v}.
\end{aligned}$$

Corollary 1. *The vector fields $Y_1, \dots, Y_{14} \in \text{Vec}(\mathbb{R}^5)$ given in Theorem 6 provide a faithful representation of the Lie algebra $\mathfrak{g}_2$.*

The proof of this theorem reduces to the following two independent lemmas.

Lemma 5.1. $\text{Sym}(\Delta) = \text{span}(Y_1, \dots, Y_{14})$.

Lemma 5.2. $\text{span}(Y_1, \dots, Y_{14}) \cong \mathfrak{g}_2$.

Proof of Lemma 5.1. We take an arbitrary smooth vector field

$$Y = P \frac{\partial}{\partial x} + Q \frac{\partial}{\partial y} + R \frac{\partial}{\partial z} + S \frac{\partial}{\partial u} + T \frac{\partial}{\partial v} \in \text{Vec}(\mathbb{R}^5),$$

and compute the brackets

$$\begin{aligned} [\xi_1, Y] &= P_x \frac{\partial}{\partial x} + Q_x \frac{\partial}{\partial y} + R_x \frac{\partial}{\partial z} + S_x \frac{\partial}{\partial u} + T_x \frac{\partial}{\partial v}, \\ [\xi_2, Y] &= E_P \frac{\partial}{\partial x} + E_Q \frac{\partial}{\partial y} + (E_R - P) \frac{\partial}{\partial z} + (E_S - R) \frac{\partial}{\partial u} + (E_T - 2xP) \frac{\partial}{\partial v}, \end{aligned}$$

where

$$(5.8) \quad E_P = \xi_2 P = P_y + xP_z + zP_u + x^2P_v,$$

$$(5.9) \quad E_Q = \xi_2 Q = Q_y + xQ_z + zQ_u + x^2Q_v,$$

$$E_R = \xi_2 R = R_y + xR_z + zR_u + x^2R_v,$$

$$E_S = \xi_2 S = S_y + xS_z + zS_u + x^2S_v,$$

$$E_T = \xi_2 T = T_y + xT_z + zT_u + x^2T_v.$$

By Proposition 1, a vector field Y is a symmetry of the distribution Δ iff

$$(5.10) \quad \begin{pmatrix} P_x \\ Q_x \\ R_x \\ S_x \\ T_x \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \\ x \\ z \\ x^2 \end{pmatrix}, \quad \begin{pmatrix} E_P \\ E_Q \\ E_R - P \\ E_S - R \\ E_T - 2xP \end{pmatrix} = \gamma \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \delta \begin{pmatrix} 0 \\ 1 \\ x \\ z \\ x^2 \end{pmatrix}$$

for some smooth real-valued functions $\alpha, \beta, \gamma, \delta$.

These vector equations are solvable in $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ iff the following equalities hold:

$$(5.11) \quad R_x = xQ_x,$$

$$(5.12) \quad S_x = zQ_x,$$

$$(5.13) \quad T_x = x^2Q_x,$$

$$(5.14) \quad P = E_R - xE_Q,$$

$$(5.15) \quad R = E_S - zE_Q,$$

$$(5.16) \quad x^2E_Q - 2xE_R + E_T = 0.$$

Equality (5.12) is obviously equivalent to

$$(5.17) \quad S = zQ + \varphi$$

for some smooth function

$$\varphi = \varphi(y, z, u, v).$$

Now we can compute R in terms of Q and φ :

$$\begin{aligned} E_S &= \xi_2 S = \xi_2(zQ + \varphi) = z\xi_2 Q + xQ + \xi_2 \varphi \\ &= zE_Q + xQ + \varphi_y + x\varphi_z + z\varphi_u + x^2\varphi_v; \end{aligned}$$

hence by (5.15),

$$(5.18) \quad R = E_S - zE_Q = xQ + \varphi_y + x\varphi_z + z\varphi_u + x^2\varphi_v.$$

We differentiate the previous equality with respect to x :

$$R_x = xQ_x + Q + \varphi_z + 2x\varphi_v,$$

and in view of (5.11) obtain

$$(5.19) \quad Q = -\varphi_z - 2x\varphi_v.$$

Then (5.17) is rewritten in the form

$$(5.20) \quad S = zQ + \varphi = \varphi - z\varphi_z - 2xz\varphi_v$$

and (5.13) as

$$T_x = x^2Q_x = -2x^2\varphi_v.$$

We integrate the previous equation with respect to x :

$$(5.21) \quad T = -\frac{2}{3}x^3\varphi_v + \psi, \quad \psi = \psi(y, z, u, v).$$

Taking into account equalities (5.19), (5.18), (5.20), and (5.21), we see that conditions (5.11)–(5.16) are equivalent to the following ones:

$$(5.22) \quad P = E_R - xE_Q,$$

$$(5.23) \quad Q = -\varphi_z - 2x\varphi_v,$$

$$(5.24) \quad R = \varphi_y + z\varphi_u - x^2\varphi_v,$$

$$(5.25) \quad S = \varphi - z\varphi_z - 2xz\varphi_v,$$

$$(5.26) \quad T = -\frac{2}{3}x^3\varphi_v + \psi,$$

$$(5.27) \quad x^2E_Q - 2xE_R + E_T = 0.$$

Thus all components of our vector field P, Q, R, S, T are uniquely determined by two functions $\varphi = \varphi(y, z, u, v)$ and $\psi = \psi(y, z, u, v)$ that satisfy equation (5.27). Now we substitute the expressions of P, Q, R, S, T in terms of φ, ψ into this equation and find independent parameters that determine φ, ψ .

In terms of φ, ψ , equality (5.27) takes the form

$$\begin{aligned} &x^5 \left(-\frac{2}{3}\varphi_{vv} \right) + x^4 \left(-\frac{5}{3}\varphi_{zv} \right) + x^3 \left(-\frac{8}{3}\varphi_{yv} - \varphi_{zz} - \frac{8}{3}z\varphi_{uv} \right) \\ &+ x^2(-3\varphi_{yz} - 3z\varphi_{zu} - 2\varphi_u + \psi_v) \\ &+ x(-2\varphi_{yy} - 4z\varphi_{yu} - 2z^2\varphi_{uu} + \psi_z) + (\psi_y + z\psi_u) = 0. \end{aligned}$$

Recall that φ does not depend on x ; that is why

$$(5.28) \quad \varphi_{vv} = 0,$$

$$(5.29) \quad \varphi_{zv} = 0,$$

$$(5.30) \quad \varphi_{yv} + \frac{3}{8}\varphi_{zz} + z\varphi_{uv} = 0,$$

$$(5.31) \quad 3\varphi_{yz} + 3z\varphi_{zu} + 2\varphi_u - \psi_v = 0,$$

$$(5.32) \quad 2\varphi_{yy} + 4z\varphi_{yu} + 2z^2\varphi_{uu} - \psi_z = 0,$$

$$(5.33) \quad \psi_y + z\psi_u = 0.$$

Equations (5.28), (5.29) mean that φ_v does not depend on v, z :

$$\varphi_v = \alpha(y, u);$$

thus,

$$(5.34) \quad \varphi = v\alpha(y, u) + \beta(y, z, u)$$

for some functions $\alpha(y, u)$ and $\beta(y, z, u)$. In view of the previous equality, conditions (5.30)–(5.33) take the form

$$(5.35) \quad \alpha_y + \frac{3}{8}\beta_{zz} + z\alpha_u = 0,$$

$$(5.36) \quad 3\beta_{yz} + 3z\beta_{zu} + 2v\alpha_u + 2\beta_u - \psi_v = 0,$$

$$(5.37) \quad 2v\alpha_{yy} + 2\beta_{yy} + 4zv\alpha_{yu} + 4z\beta_{yu} + 2z^2v\alpha_{uu} + 2z^2\beta_{uu} - \psi_z = 0,$$

$$(5.38) \quad \psi_y + z\psi_u = 0.$$

Differentiation of equation (5.35) with respect to z gives

$$\frac{3}{8}\beta_{zzz} = -\alpha_u.$$

Then we integrate the previous equality three times with respect to z and obtain

$$(5.39) \quad \beta = -\frac{4}{9}z^3\alpha_u + \frac{1}{2}z^2\gamma + z\delta + \sigma$$

for some functions

$$\gamma = \gamma(y, u), \quad \delta = \delta(y, u), \quad \sigma = \sigma(y, u).$$

We substitute expression (5.39) into (5.35) and obtain

$$\gamma = -\frac{8}{3}\alpha_y;$$

thus,

$$(5.40) \quad \beta = -\frac{4}{9}z^3\alpha_u - \frac{4}{3}z^2\alpha_y + z\delta + \sigma,$$

and this equality is equivalent to (5.35).

Substitution of the expression for β from (5.40) to equalities (5.36) and (5.37) leads, after some transformations, to

$$\begin{aligned} \psi_v &= z^3 \left(-\frac{44}{9} \alpha_{uu} \right) + z^2 \left(-\frac{44}{3} \alpha_{yu} \right) + z(-8\alpha_{yy} + 5\delta_u) \\ &\quad + (3\delta_y + 2v\alpha_u + 2\sigma_u), \end{aligned} \quad (5.41)$$

$$\begin{aligned} \psi_z &= z^5 \left(-\frac{8}{9} \alpha_{uuu} \right) + z^4 \left(-\frac{40}{9} \alpha_{yuu} \right) + z^3 \left(-\frac{56}{9} \alpha_{yyu} + 2\delta_{uu} \right) \\ &\quad + z^2 \left(-\frac{8}{3} \alpha_{yyy} + 4\delta_{yu} + 2v\alpha_{uu} + 2\sigma_{uu} \right) \\ &\quad + z(2\delta_{yy} + 4v\alpha_{yu} + 4\sigma_{yu}) + (2v\alpha_{yy} + 2\sigma_{yy}). \end{aligned} \quad (5.42)$$

We differentiate:

$$\begin{aligned} \psi_{vz} &= -\frac{44}{3} z^2 \alpha_{uu} + 2z \left(-\frac{44}{3} \alpha_{yu} \right) + (-8\alpha_{yy} + 5\delta_u), \\ \psi_{zv} &= 2z^2 \alpha_{uu} + 4z \alpha_{yu} + 2\alpha_{yy}, \end{aligned}$$

equate these mixed derivatives and the terms near equal powers of z in them, and obtain

$$(5.43) \quad \alpha_{uu} = 0,$$

$$(5.44) \quad \alpha_{yu} = 0,$$

$$(5.45) \quad \alpha_{yy} = \frac{1}{2} \delta_u.$$

Equations (5.43) and (5.44) mean that

$$\alpha_u = c, \quad c \in \mathbb{R};$$

thus,

$$(5.46) \quad \alpha = cu + \pi(y).$$

Condition (5.45) then reads

$$\pi_{yy} = \frac{1}{2} \delta_u;$$

that is why

$$(5.47) \quad \delta = 2u\pi_{yy} + \lambda(y).$$

In view of (5.46) and (5.47), equalities (5.41) and (5.42) are rewritten as

$$(5.48) \quad \psi_v = z(2\pi_{yy}) + 6u\pi_{yyy} + 3\lambda_y + 2cv + 2\sigma_u,$$

$$\begin{aligned} \psi_z &= z^2 \left(\frac{16}{3} \pi_y^{(3)} + 2\sigma_{uu} \right) + z(4u\pi_y^{(4)} + 2\lambda_{yy} + 4\sigma_{yu}) \\ &\quad + (2v\pi_{yy} + 2\sigma_{yy}), \end{aligned} \quad (5.49)$$

the first of which after integration with respect to v gives

$$(5.50) \quad \psi = cv^2 + v(2z\pi_{yy} + 6u\pi_{yyy} + 3\lambda_y + 2\sigma_u) + \tau(y, z, u).$$

We differentiate the previous equality with respect to z and obtain

$$\psi_z = 2v\pi_{yy} + \tau_z,$$

then compare with (5.49) and get

$$\tau_z = z^2 \left(\frac{16}{3} \pi_y^{(3)} + 2\sigma_{uu} \right) + z(4u\pi_y^{(4)} + 2\lambda_{yy} + 4\sigma_{yu}) + 2\sigma_{yy}.$$

Integrating with respect to z leads to

$$\tau = z^3 \left(\frac{16}{9} \pi_y^{(3)} + \frac{2}{3} \sigma_{uu} \right) + z^2(2u\pi_y^{(4)} + \lambda_{yy} + 2\sigma_{yu}) + 2\sigma_{yy}z + \varepsilon(y, u).$$

We substitute this into (5.50) and obtain

$$\begin{aligned} \psi &= cv^2 + v(2z\pi_y^{(2)} + 6u\pi_y^{(3)} + 3\lambda_y + 2\sigma_u) + z^3 \left(\frac{16}{9} \pi_y^{(3)} + \frac{2}{3} \sigma_{uu} \right) \\ (5.51) \quad &+ z^2(2u\pi_y^{(4)} + \lambda_{yy} + 2\sigma_{yu}) + 2\sigma_{yy}z + \varepsilon. \end{aligned}$$

We substitute this expression for ψ into (5.38) and get

$$\begin{aligned} &z^4 \left(\frac{2}{3} \sigma_{uuu} \right) + z^3 \left(\frac{34}{9} \pi_y^{(4)} + \frac{8}{3} \sigma_{yuu} \right) + z^2(2u\pi_y^{(5)} + \lambda_y^{(3)} + 4\sigma_{yyu}) \\ &+ zv(8\pi_y^{(3)} + 2\sigma_{uu}) + z(2\sigma_{yyy} + \varepsilon_u) + v(6u\pi_y^{(4)} + 3\lambda_y^{(2)} + 2\sigma_{yu}) + \varepsilon_y = 0. \end{aligned}$$

We equate terms near powers of z and v to zero and obtain

$$(5.52) \quad \sigma_{uuu} = 0,$$

$$(5.53) \quad \frac{34}{9} \pi_y^{(4)} + \frac{8}{3} \sigma_{yuu} = 0,$$

$$(5.54) \quad 2u\pi_y^{(5)} + \lambda_y^{(3)} + 4\sigma_{yyu} = 0,$$

$$(5.55) \quad 8\pi_y^{(3)} + 2\sigma_{uu} = 0,$$

$$(5.56) \quad 2\sigma_{yyy} + \varepsilon_u = 0,$$

$$(5.57) \quad 6u\pi_y^{(4)} + 3\lambda_y^{(2)} + 2\sigma_{yu} = 0,$$

$$(5.58) \quad \varepsilon_y = 0.$$

Equation (5.58) means that

$$\varepsilon = \varepsilon(u),$$

and equation (5.53) yields

$$(5.59) \quad \sigma = \frac{1}{2}u^2\theta(y) + u\rho(y) + \gamma(y).$$

Then equalities (5.53)–(5.58) are rewritten as

$$(5.60) \quad \frac{34}{9} \pi_y^{(4)} + \frac{8}{3} \theta_y = 0,$$

$$(5.61) \quad 2u\pi_y^{(5)} + \lambda_y^{(3)} + 4u\theta_y^{(2)} + 4\rho_y^{(2)} = 0,$$

$$(5.62) \quad 8\pi_y^{(3)} + 2\theta = 0,$$

$$(5.63) \quad u^2\theta_y^{(3)} + 2u\rho_y^{(3)} + 2\gamma_y^{(3)} + \varepsilon_u = 0,$$

$$(5.64) \quad 6u\pi_y^{(4)} + 3\lambda_y^{(2)} + 2u\theta_y + 2\rho_y = 0.$$

We differentiate (5.62):

$$8\pi_y^{(4)} + 2\theta_y = 0,$$

which gives in combination with (5.60):

$$\pi_y^{(4)} = \theta_y = 0.$$

Thus,

$$(5.65) \quad \theta = a, \quad a \in \mathbb{R},$$

and equations (5.60)–(5.64) take the form

$$(5.66) \quad \lambda_y^{(3)} + 4\rho_y^{(2)} = 0,$$

$$(5.67) \quad 8\pi_y^{(3)} + 2a = 0,$$

$$(5.68) \quad 2u\rho_y^{(3)} + 2\gamma_y^{(3)} + \varepsilon_u = 0,$$

$$(5.69) \quad 3\lambda_y^{(2)} + 2\rho_y = 0.$$

Then (5.67) implies

$$(5.70) \quad \pi = -\frac{1}{24}ay^3 + by^2 + dy + f, \quad b, d, f \in \mathbb{R},$$

and equations (5.66)–(5.69) are equivalent to

$$\begin{aligned} \lambda_y^{(3)} &= 0, \\ \rho_y^{(2)} &= 0, \\ 2\gamma_y^{(3)} + \varepsilon_y &= 0, \\ 3\lambda_y^{(2)} + 2\rho_y &= 0. \end{aligned}$$

That is why

$$\begin{aligned} \lambda &= -\frac{1}{3}ny^2 + ly + m, \\ \rho &= ny + p, \\ \gamma &= \frac{1}{12}ky^3 + qy^2 + ry + s, \\ \varepsilon &= -ku + t \end{aligned}$$

for some

$$n, l, m, p, k, q, r, s, t \in \mathbb{R}.$$

Now we recover the functions φ and ψ via (5.34), (5.40), (5.46), (5.47), (5.51), (5.59), (5.65), and (5.70):

$$\begin{aligned} \varphi &= \frac{1}{6}ay^2z^2 - \frac{1}{24}ay^3v + by^2v - \frac{4}{9}cz^3 - \frac{8}{3}byz^2 - \frac{1}{2}ayzu \\ &\quad - \frac{1}{3}ny^2z + \frac{1}{12}ky^3 + cuv + dyv - \frac{4}{3}dz^2 + 4bzu + lyz + \frac{1}{2}au^2 \\ &\quad + nyu + qy^2 + fv + mz + pu + ry + s, \\ \psi &= -\frac{1}{2}ayzv + \frac{2}{9}az^3 + cv^2 + 4bzv + \frac{1}{2}auv + \frac{4}{3}nz^2 + kyz + 3lv \\ &\quad + 2pv + 4qz - ku + t, \end{aligned}$$

$$a, b, c, d, f, n, l, m, p, k, q, r, s, t \in \mathbb{R}.$$

To summarize, we proved that

$$\text{Sym}(\Delta) \subset \left\{ Y = P \frac{\partial}{\partial x} + Q \frac{\partial}{\partial y} + R \frac{\partial}{\partial z} + S \frac{\partial}{\partial u} + T \frac{\partial}{\partial v} \right\},$$

where

$$(5.71) \quad \begin{aligned} P = & \frac{1}{12}ax^2y^2 - \frac{4}{3}bx^2y - \frac{2}{3}cx^2z - \frac{1}{6}axyz - \frac{2}{3}dx^2 - \frac{1}{3}nxy + \frac{4}{3}bxz \\ & + cxv + \frac{1}{3}az^2 - \frac{1}{4}ayv + (p+2l)x + \frac{1}{2}ky + \frac{4}{3}nz + 2bv + 2q, \end{aligned}$$

$$(5.72) \quad \begin{aligned} Q = & \frac{1}{12}axy^3 - 2bxy^2 - \frac{1}{3}azy^2 - 2dxy - 2cxu + \frac{4}{3}cz^2 + \frac{16}{3}byz + \frac{1}{3}ny^2 \\ & + \frac{1}{2}ayu - 2fx - ly + \frac{8}{3}dz - 4bu - m, \end{aligned}$$

$$(5.73) \quad \begin{aligned} R = & \frac{1}{24}ax^2y^3 - bx^2y^2 - dx^2y - cx^2u - \frac{1}{6}ayz^2 - \frac{1}{8}ay^2v - fx^2 + \frac{1}{3}nyz \\ & + 2byz + \frac{4}{3}bz^2 + \frac{1}{2}azu + \frac{1}{4}ky^2 + czv + 2qy + lz + nu \\ & + dv + pz + r, \end{aligned}$$

$$(5.74) \quad \begin{aligned} S = & \frac{1}{12}axy^3z - 2bxy^2z - \frac{1}{6}ay^2z^2 - \frac{1}{24}avy^3 - 2cxzu - 2dxyz + by^2v \\ & + \frac{8}{3}byz^2 + \frac{1}{12}ky^3 + \frac{8}{9}cz^3 - 2fxz + qy^2 + nyu + \frac{1}{2}au^2 + cuv + dyv \\ & + \frac{4}{3}dz^2 + ry + pu + fv + s, \end{aligned}$$

$$(5.75) \quad \begin{aligned} T = & \frac{1}{36}ax^3y^3 - \frac{2}{3}bx^3y^2 - \frac{2}{3}dx^3y - \frac{2}{3}cx^3u - \frac{2}{3}fx^3 - \frac{1}{2}ayzv + \frac{2}{9}az^3 \\ & + \frac{4}{3}nz^2 + kyz + 4bzu + \frac{1}{2}auv + cv^2 + 4qz - ku + (3l+2p)v + t. \end{aligned}$$

The basis of the Lie algebra $\text{Sym}(\Delta)$ presented in the formulation of Theorem 6 is obtained for the following values of parameters (we write only nonzero values of the parameters $k, q, m, r, s, t, a, b, c, d, f, n, p, l$):

$$(5.76) \quad q = \frac{1}{36} \Rightarrow Y_1,$$

$$(5.77) \quad m = 3 \Rightarrow Y_2,$$

$$(5.78) \quad r = \frac{1}{12} \Rightarrow Y_3,$$

$$(5.79) \quad t = -\frac{1}{324} \Rightarrow Y_4,$$

$$(5.80) \quad s = \frac{1}{12} \Rightarrow Y_5,$$

$$a = 24 \Rightarrow Y_6,$$

$$b = 9 \Rightarrow Y_7,$$

$$c = -324 \Rightarrow Y_8,$$

$$d = -27 \Rightarrow Y_9,$$

$$n = -1 \Rightarrow Y_{10},$$

$$f = 27 \Rightarrow Y_{11},$$

$$k = \frac{1}{27} \Rightarrow Y_{12},$$

$$p = -1 \Rightarrow Y_{13},$$

$$p = 1, l = -1 \Rightarrow Y_{14}.$$

Immediate verification shows that the vector fields $Y_1, \dots, Y_{14}$ are linearly independent.

All vector fields $Y_1, \dots, Y_{14}$ are indeed symmetries of the distribution Δ since the conditions of Proposition 1 hold:

$$\begin{array}{ll}
[Y_i, \xi_1] = 0, & [Y_i, \xi_2] = 0, \quad i = 1, \dots, 5, \\
[Y_6, \xi_1] = (-4xy^2 + 4yz)\xi_1 & [Y_6, \xi_2] = (6x^2y - 12xz + 6v)\xi_1 \\
& \quad - 2y^3\xi_2, \\
[Y_7, \xi_1] = (24xy - 12z)\xi_1 & [Y_7, \xi_2] = -18x^2\xi_1 \\
& \quad + 18y^2\xi_2, \\
[Y_8, \xi_1] = (-432xz + 324v)\xi_1 & [Y_8, \xi_2] = 108x^3\xi_1 \\
& \quad - 648u\xi_2, \\
[Y_9, \xi_1] = -36x\xi_1 - 54y\xi_2, & [Y_9, \xi_2] = 18x\xi_2, \\
[Y_{10}, \xi_1] = -\frac{1}{3}y\xi_1, & [Y_{10}, \xi_2] = x\xi_1 + \frac{2}{3}y\xi_2, \\
[Y_{11}, \xi_1] = 54\xi_2, & [Y_{11}, \xi_2] = 0, \\
[Y_{12}, \xi_1] = 0, & [Y_{12}, \xi_2] = -\frac{1}{54}\xi_1, \\
[Y_{13}, \xi_1] = \xi_1, & [Y_{13}, \xi_2] = 0, \\
[Y_{14}, \xi_1] = \xi_1, & [Y_{14}, \xi_2] = -\xi_2.
\end{array}$$

Thus,

$$\text{Sym}(\Delta) = \text{span}(Y_1, \dots, Y_{14}),$$

and Lemma 5.1 is completely proved. $\square$

Proof of Lemma 5.2. Nonzero brackets in the Lie algebra $\text{Sym}(\Delta)$ in the basis $Y_1, \dots, Y_{14}$ given in the formulation of Theorem 6 are shown in Table 1.

TABLE 1. Multiplication in $\text{Sym}(\Delta)$, the Cartan case

$$\begin{array}{lll}
[Y_3, Y_{10}] = -2Y_1, & [Y_3, Y_9] = 2Y_2, & [Y_3, Y_2] = 3Y_5, \\
[Y_3, Y_1] = -3Y_4, & [Y_3, Y_8] = Y_9, & [Y_3, Y_6] = -Y_{10}, \\
[Y_{10}, Y_9] = -2Y_7, & [Y_{10}, Y_7] = -3Y_6 & [Y_{10}, Y_1] = 3Y_{12}, \\
[Y_{10}, Y_5] = Y_3, & [Y_{10}, Y_{11}] = -Y_9, & [Y_9, Y_7] = 3Y_8, \\
[Y_9, Y_2] = -3Y_{11}, & [Y_9, Y_{12}] = Y_{10}, & [Y_9, Y_4] = -Y_3, \\
[Y_7, Y_2] = -2Y_9 & [Y_7, Y_1] = 2Y_{10}, & [Y_7, Y_5] = -Y_2, \\
[Y_7, Y_4] = Y_1, & [Y_2, Y_1] = -2Y_3 & [Y_2, Y_{12}] = -Y_1, \\
[Y_2, Y_6] = Y_7, & [Y_1, Y_8] = -Y_7, & [Y_1, Y_{11}] = Y_2, \\
[Y_{12}, Y_8] = Y_6, & [Y_{12}, Y_5] = -Y_4, & [Y_8, Y_5] = Y_{11}, \\
[Y_{11}, Y_4] = Y_5 & [Y_{11}, Y_6] = -Y_8, & [Y_4, Y_6] = Y_{12}, \\
[Y_{13}, Y_3] = Y_3, & [Y_{13}, Y_9] = -Y_9 & [Y_{13}, Y_7] = -Y_7, \\
[Y_{13}, Y_1] = Y_1, & [Y_{13}, Y_{12}] = Y_{12}, & [Y_{13}, Y_8] = -2Y_8, \\
[Y_{13}, Y_5] = Y_5, & [Y_{13}, Y_{11}] = -Y_{11}, & [Y_{13}, Y_4] = 2Y_4, \\
[Y_{13}, Y_6] = -Y_6, & [Y_{14}, Y_{10}] = Y_{10}, & [Y_{14}, Y_9] = -Y_9, \\
[Y_{14}, Y_2] = -Y_2, & [Y_{14}, Y_1] = Y_1, & [Y_{14}, Y_{12}] = 2Y_{12}, \\
[Y_{14}, Y_8] = -Y_8, & [Y_{14}, Y_5] = -Y_5, & [Y_{14}, Y_{11}] = -2Y_{11}, \\
[Y_{14}, Y_4] = Y_4, & [Y_{14}, Y_6] = Y_6, & \\
[Y_3, Y_7] = -2Y_{13} + Y_{14}, & [Y_{10}, Y_2] = Y_{13} - 2Y_{14}, & \\
[Y_9, Y_1] = Y_{13} + Y_{14}, & [Y_{12}, Y_{11}] = -Y_{14}, & \\
[Y_8, Y_4] = Y_{13}, & [Y_5, Y_6] = -Y_{13} + Y_{14}. &
\end{array}$$

The required isomorphism

$$F : \text{Sym}(\Delta) \rightarrow \mathfrak{g}_2$$

is defined on the bases of these Lie algebras by the following matrix:

$Y \in \text{Sym}(\Delta)$	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6	Y_7
$F(Y) \in \mathfrak{g}_2$	X_{-e_3}	X_{-e_2}	X_{e_1}	X_{-f_2}	X_{f_3}	X_{-f_3}	X_{-e_1}

$Y \in \text{Sym}(\Delta)$	Y_8	Y_9	Y_{10}	Y_{11}	Y_{12}	Y_{13}	Y_{14}
$F(Y) \in \mathfrak{g}_2$	X_{f_2}	X_{e_3}	X_{e_2}	X_{-f_1}	X_{f_1}	H_1	H_2

See the appendix for a description of the Lie algebra $\mathfrak{g}_2$ and its basis $H_1, H_2, X_{\pm e_i}, X_{\pm f_i}, i = 1, 2, 3$. The map F is indeed an isomorphism of $\text{Sym}(\Delta)$ and $\mathfrak{g}_2$ since the multiplication Tables 1 and 3 (see the appendix) for these Lie algebras are isomorphic. $\square$

5.4.2. Symmetries of the sub-Riemannian structure.

Theorem 7. *The Lie algebra of symmetries of the flat sub-Riemannian structure $(\Delta, \langle \cdot, \cdot \rangle)$ on the five-dimensional nilpotent Lie group G described in Subsection 5.1 is the six-dimensional Lie algebra*

$$\text{Sym}(\Delta, \langle \cdot, \cdot \rangle) = \text{span}(X_0, \dots, X_5)$$

with the following multiplication rules for the basis elements:

$$(5.81) \quad \begin{aligned} [X_0, X_1] &= -X_2, & [X_0, X_2] &= X_1, \\ [X_0, X_4] &= -X_5, & [X_0, X_5] &= X_4, \\ [X_1, X_2] &= X_3, \\ [X_1, X_3] &= X_4, & [X_2, X_3] &= X_5. \end{aligned}$$

For the model of $(\Delta, \langle \cdot, \cdot \rangle)$ in $\mathbb{R}^5$ defined by (5.4)–(5.7) we have

$$(5.82) \quad X_0 = -\frac{1}{54}Y_{11} - 54Y_{12}$$

and

$$(5.83) \quad X_1 = -3Y_1, \quad X_2 = \frac{1}{18}Y_2, \quad X_3 = -\frac{1}{3}Y_3, \quad X_4 = 3Y_4, \quad X_5 = \frac{1}{18}Y_5,$$

where the vector fields $Y_1, \dots, Y_5$ are defined in Theorem 6.

Remark 5. Multiplication rules (5.81) in the Lie algebra $\text{Sym}(\Delta, \langle \cdot, \cdot \rangle)$ are schematically represented in Figure 7 (we draw X_0 twice to obtain a planar graph).

Proof. By the proof of Theorem 6, a vector field

$$X = P \frac{\partial}{\partial x} + Q \frac{\partial}{\partial y} + R \frac{\partial}{\partial z} + S \frac{\partial}{\partial u} + T \frac{\partial}{\partial v} \in \text{Vec}(\mathbb{R}^5)$$

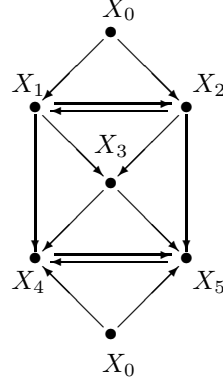
is a symmetry of the distribution Δ iff the functions P, Q, R, S, T have the form (5.71)–(5.75). Moreover, X also preserves the inner product $\langle \cdot, \cdot \rangle$ iff P, Q, R, S, T satisfy, in addition to (5.71)–(5.75), the following extra equations:

$$(5.84) \quad P_x = 0,$$

$$(5.85) \quad Q_x = -E_P,$$

$$(5.86) \quad E_Q = 0$$

(the notation E_P, E_Q is introduced in (5.8), (5.9)).

FIGURE 7. $\text{Sym}(\Delta, \langle \cdot, \cdot \rangle)$, the Cartan case

But conditions (5.84)–(5.86) mean that

$$a = b = c = d = n = p = l = 0, \quad f = \frac{1}{4}k,$$

i.e.,

$$\begin{aligned} P &= \frac{1}{2}ky + 2q, \\ Q &= -\frac{1}{2}kx - m, \\ R &= -\frac{1}{4}kx^2 + \frac{1}{4}ky^2 + 2qy + r, \\ S &= \frac{1}{12}ky^3 - \frac{1}{2}kxz + qy^2 + ry + \frac{1}{4}kv + s, \\ T &= -\frac{1}{6}kx^3 + kyz + 4qz - ku + t. \end{aligned}$$

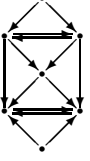
For $k = -2$, $q = m = r = s = t = 0$ we obtain the vector field X_0 defined by (5.82). The remaining basis vector fields X_i , $i = 1, \dots, 5$, are determined by (5.83), (5.76). Then $\text{Sym}(\Delta, \langle \cdot, \cdot \rangle) = \text{span}(X_0, \dots, X_5)$, and the commutation rules (5.81) are directly verified. $\square$

6. GENERAL PICTURE

We summarize the above computations of symmetries of flat rank-two distributions and sub-Riemannian structures in Table 2.

For completeness, we include the two-dimensional (Riemannian) case: any vector field in $G = \mathbb{R}^2$ preserves the rank-two distribution TG , and the flat Riemannian structure is preserved by the Euclidean group of motions of the plane. In the Heisenberg case, $n = 3$, symmetries of flat distributions are parametrized by arbitrary smooth functions of three variables, and the flat sub-Riemannian structure is preserved by the four-dimensional Lie algebra: in addition to three independent left translations on the Heisenberg group, there is one additional rotation in

TABLE 2. Symmetries of flat $(2, n)$ -distributions and sub-Riemannian structures

n	$(\Delta, \langle \cdot, \cdot \rangle)$	$\text{Sym}(\Delta)$	$\dim$	$\text{Sym}(\Delta, \langle \cdot, \cdot \rangle)$	$\dim$
2		$\text{Vec}(\mathbb{R}^2)$	∞		3
3		$f(x, y, z)$	∞		4
4		$f(y, z, v)$	∞		4
5		$\mathfrak{g}_2$	14		6

this group. In the Engel case, $n = 4$, the Lie algebra $\text{Sym}(\Delta)$ is parametrized by functions of four variables constant along the canonical vector field of the Engel distribution (which is here taken to be $\xi_1 = \frac{\partial}{\partial x}$ as in the model of Subsection 4.3). As for symmetries of the flat sub-Riemannian structure, there is only the “trivial” four-dimensional group of left translations on the Engel group. In the Cartan case, $n = 5$, there is the 14-dimensional Lie algebra $\mathfrak{g}_2$ of symmetries of the flat distribution; and the flat sub-Riemannian structure is preserved by five “trivial” left translations on the 5-dimensional nilpotent Lie group and one additional rotation on this group.

7. APPENDIX: LINEAR REPRESENTATION OF $\mathfrak{g}_2$

For completeness of exposition, we describe here a faithful representation of the simple exceptional complex Lie algebra $\mathfrak{g}_2^{\mathbb{C}}$ and its unique noncompact real form by 7×7 complex skew-symmetric matrices (we follow [11], Lecture 14).

We denote by

$$E_{ij}, \quad i, j = 1, \dots, 7,$$

the 7×7 matrix with all zero entries except the only identity entry in the i -th row and j -th column; introduce also the skew-symmetric matrices — the basis elements of the Lie algebra $\mathfrak{so}(7)$:

$$E_{[i,j]} = \frac{1}{2}(E_{ij} - E_{ji}), \quad i, j = 1, \dots, 7, \quad i < j.$$

Then

$$\mathfrak{g}_2^{\mathbb{C}} = \text{span}_{\mathbb{C}}(P_0, \dots, P_6, Q_0, \dots, Q_6),$$

where

$$\begin{aligned} P_0 &= 2(E_{[3,2]} + E_{[6,7]}), & Q_0 &= 2(E_{[4,5]} + E_{[6,7]}), \\ P_1 &= E_{[1,3]} + E_{[5,7]}, & Q_1 &= E_{[6,4]} + E_{[5,7]}, \\ P_2 &= E_{[2,1]} + E_{[7,4]}, & Q_2 &= E_{[6,5]} + E_{[7,4]}, \\ P_3 &= E_{[1,4]} + E_{[7,2]}, & Q_3 &= E_{[3,6]} + E_{[7,2]}, \\ P_4 &= E_{[5,1]} + E_{[3,7]}, & Q_4 &= E_{[2,6]} + E_{[3,7]}, \\ P_5 &= E_{[1,7]} + E_{[3,5]}, & Q_5 &= E_{[4,2]} + E_{[3,5]}, \\ P_6 &= E_{[6,1]} + E_{[4,3]}, & Q_6 &= E_{[5,2]} + E_{[4,3]}. \end{aligned}$$

Matrices of the form

$$(7.1) \quad H = aP_0 + bQ_0 = aE_{[3,2]} + bE_{[4,5]} + cE_{[7,6]}, \quad a + b + c = 0,$$

form a two-dimensional Abelian subalgebra $\mathfrak{h}$ of $\mathfrak{g}_2^{\mathbb{C}}$ ($\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}_2^{\mathbb{C}}$).

Introduce the elements

$$\begin{aligned} U_{\pm 1} &= (2P_2 - Q_2) \pm i(2P_1 - Q_1), & V_{\pm 1} &= Q_2 \mp iQ_1, \\ U_{\pm 2} &= (2P_4 - Q_4) \pm i(2P_3 - Q_3), & V_{\pm 2} &= Q_4 \pm iQ_3, \\ U_{\pm 3} &= (2P_6 - Q_6) \pm i(2P_5 - Q_5), & V_{\pm 3} &= Q_6 \pm iQ_5, \end{aligned}$$

which span $\mathfrak{g}_2^{\mathbb{C}}$ together with $\mathfrak{h}$.

In the dual space $\mathfrak{h}^*$ we choose the basis e_1, e_2 dual to the basis P_0, Q_0 . Then for each element (7.1) of the space $\mathfrak{h}$ we have

$$e_1(H) = a, \quad e_2(H) = b, \quad e_3(H) = c,$$

where

$$e_3 = -(e_1 + e_2).$$

Assume that the dual space $\mathfrak{h}^*$ is a Euclidean space with Cartesian coordinates in which the vectors e_1, e_2 have the coordinates

$$e_1 = \left(\frac{\sqrt{6}}{3}, 0 \right), \quad e_2 = \left(-\frac{\sqrt{6}}{6}, \frac{\sqrt{2}}{2} \right).$$

Introduce also the vectors

$$f_1 = e_2 - e_3, \quad f_2 = e_3 - e_3, \quad f_3 = e_1 - e_2.$$

Then we obtain the following 12 vectors in the plane $\mathfrak{h}^*$:

$$\mathbf{G}_2 = \{ \pm e_1, \pm e_2, \pm e_3, \pm f_1, \pm f_2, \pm f_3 \};$$

see Figure 8.

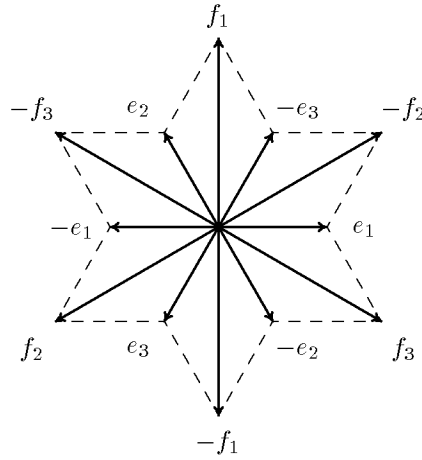
Now we choose the following elements in $\mathfrak{g}_2^{\mathbb{C}}$:

$$\begin{aligned} X_{\alpha} &= \begin{cases} U_{\pm k} & \text{if } \alpha = \pm e_k, \\ V_{\pm k} & \text{if } \alpha = \pm f_k, \end{cases} \\ H_{\alpha} &= \frac{2}{a^2 + b^2 + c^2} (aE_{[3,2]} + bE_{[4,5]} + cE_{[7,6]}). \end{aligned}$$

Then

$$\mathfrak{g}_2^{\mathbb{C}} = \text{span}(\mathfrak{h}; X_{\alpha}, \alpha \in \mathbf{G}_2)$$

and multiplication rules in the Lie algebra $\mathfrak{g}_2^{\mathbb{C}}$ take the following simple form.

FIGURE 8. Root vectors of $\mathfrak{g}_2^{\mathbb{C}}$

Proposition 2 ([11]). *For any vectors $\alpha, \beta \in \mathbf{G}_2$, the following relations hold:*

$$\begin{aligned} [H, X_\alpha] &= i\alpha(H)X_\alpha, \quad H \in \mathfrak{h}, \\ [X_\alpha, X_{-\alpha}] &= iH_\alpha, \\ [X_\alpha, X_\beta] &= 0 \quad \text{if } \alpha + \beta \neq 0 \text{ and } \alpha + \beta \notin \mathbf{G}_2, \\ [X_\alpha, X_\beta] &= N_{\alpha,\beta}X_{\alpha+\beta} \text{ if } \alpha + \beta \in \mathbf{G}_2. \end{aligned}$$

Here $N_{\alpha,\beta}$ are some integers whose absolute values satisfy

$$|N_{\alpha,\beta}| = p + 1,$$

where p is the least integer such that for any $j = 0, 1, \dots, p$ the vector $\beta - j\alpha$ belongs to $\mathbf{G}_2$.

Remark 6. Direct computations with the matrices X_α yield the following values for the coefficients $N_{\alpha,\beta}$:

$N_{\alpha,\beta}$	e_1	e_2	e_3	$-e_1$	$-e_2$	$-e_3$	f_1	f_2	f_3	$-f_1$	$-f_2$	$-f_3$
e_1	0	-2	2		3	-3	0	1	0	0	0	-1
e_2	2	0	-2	-3		3	0	0	1	-1	0	0
e_3	-2	2	0	3	-3		1	0	0	0	-1	0
$-e_1$		3	-3	0	-2	2	0	0	-1	0	1	0
$-e_2$	-3		3	2	0	-2	-1	0	0	0	0	1
$-e_3$	3	-3		-2	2	0	0	-1	0	1	0	0
f_1	0	0	-1	0	1	0	0	1	-1		0	0
f_2	-1	0	0	0	0	1	-1	0	1	0		0
f_3	0	-1	0	1	0	0	1	-1	0	0	0	
$-f_1$	0	1	0	0	0	-1		0	0	0	1	-1
$-f_2$	0	0	1	-1	0	0	0		0	-1	0	1
$-f_3$	1	0	0	0	-1	0	0	0		1	-1	0

In fact, this table is directly extracted from the commutation relations from Table 3.

TABLE 3. Multiplication in $\mathfrak{g}_2$

$$\begin{array}{lll}
[X_{e_1}, X_{e_2}] = -2X_{-e_3}, & [X_{e_1}, X_{e_3}] = 2X_{-e_2}, & [X_{e_1}, X_{-e_2}] = 3X_{f_3}, \\
[X_{e_1}, X_{-e_3}] = -3X_{-f_2}, & [X_{e_1}, X_{f_2}] = X_{e_3}, & [X_{e_1}, X_{-f_3}] = -X_{e_2}, \\
[X_{e_2}, X_{e_3}] = -2X_{-e_1}, & [X_{e_2}, X_{-e_1}] = -3X_{-f_3}, & [X_{e_2}, X_{-e_3}] = 3X_{f_1}, \\
[X_{e_2}, X_{f_3}] = X_{e_1}, & [X_{e_2}, X_{-f_1}] = -X_{e_3}, & [X_{e_3}, X_{-e_1}] = 3X_{f_2}, \\
[X_{e_3}, X_{-e_2}] = -3X_{-f_1}, & [X_{e_3}, X_{f_1}] = X_{e_2}, & [X_{e_3}, X_{-f_2}] = -X_{e_1}, \\
[X_{-e_1}, X_{-e_2}] = -2X_{e_3}, & [X_{-e_1}, X_{-e_3}] = 2X_{e_2}, & [X_{-e_1}, X_{f_3}] = -X_{-e_2}, \\
[X_{-e_1}, X_{-f_2}] = X_{-e_3}, & [X_{-e_2}, X_{-e_3}] = -2X_{e_1}, & [X_{-e_2}, X_{f_1}] = -X_{-e_3}, \\
[X_{-e_2}, X_{-f_3}] = X_{-e_1}, & [X_{-e_3}, X_{f_2}] = -X_{-e_1}, & [X_{-e_3}, X_{-f_1}] = X_{-e_2}, \\
[X_{f_1}, X_{f_2}] = X_{-f_3}, & [X_{f_1}, X_{f_3}] = -X_{-f_2}, & [X_{f_2}, X_{f_3}] = X_{-f_1}, \\
[X_{-f_1}, X_{-f_2}] = X_{f_3}, & [X_{-f_1}, X_{-f_3}] = -X_{f_2}, & [X_{-f_2}, X_{-f_3}] = X_{f_1}, \\
[H_1, X_{e_1}] = X_{e_1}, & [H_1, X_{e_3}] = -X_{e_3}, & [H_1, X_{-e_1}] = -X_{-e_1}, \\
[H_1, X_{-e_3}] = X_{-e_3}, & [H_1, X_{f_1}] = X_{f_1}, & [H_1, X_{f_2}] = -2X_{f_2}, \\
[H_1, X_{f_3}] = X_{f_3}, & [H_1, X_{-f_1}] = -X_{-f_1}, & [H_1, X_{-f_2}] = 2X_{-f_2}, \\
[H_1, X_{-f_3}] = -X_{-f_3}, & [H_2, X_{e_2}] = X_{e_2}, & [H_2, X_{e_3}] = -X_{e_3}, \\
[H_2, X_{-e_2}] = -X_{-e_2}, & [H_2, X_{-e_3}] = X_{-e_3}, & [H_2, X_{f_1}] = 2X_{f_1}, \\
[H_2, X_{f_2}] = -X_{f_2}, & [H_2, X_{f_3}] = -X_{f_3}, & [H_2, X_{-f_1}] = -2X_{-f_1}, \\
[H_2, X_{-f_2}] = X_{-f_2}, & [H_2, X_{-f_3}] = X_{-f_3}, & \\
[X_{e_1}, X_{-e_1}] = -2H_1 + H_2, & [X_{e_2}, X_{-e_2}] = H_1 - 2H_2, & \\
[X_{e_3}, X_{-e_3}] = H_1 + H_2, & [X_{f_1}, X_{-f_1}] = -H_2, & \\
[X_{f_2}, X_{-f_2}] = H_1, & [X_{f_3}, X_{-f_3}] = -H_1 + H_2. &
\end{array}$$

The vectors

$$H_1 = -iP_0, \quad H_2 = -iQ_0$$

form a basis of the Cartan subalgebra $\mathfrak{h}$. Moreover, the elements

$$X_\alpha, \alpha \in \mathbf{G}_2; \quad H_1, H_2$$

make up a Cartan-Weyl basis of $\mathfrak{g}_2^{\mathbb{C}}$ with real structure constants. Then the set of elements of $\mathfrak{g}_2^{\mathbb{C}}$ invariant with respect to the complex conjugation relative to this basis, i.e.,

$$X \mapsto \overline{X},$$

form the (unique) real noncompact form of the complex Lie algebra $\mathfrak{g}_2^{\mathbb{C}}$ (see e.g. [14]):

$$\mathfrak{g}_2 = \{ X \in \mathfrak{g}_2^{\mathbb{C}} \mid \overline{X} = X \}.$$

We have

$$\mathfrak{g}_2 = \text{span}_{\mathbb{R}}(X_\alpha, \alpha \in \mathbf{G}_2; H_1, H_2),$$

and nonzero brackets between these basis vectors are given in Table 3.

ACKNOWLEDGMENT

The author wishes to thank Professor A. A. Agrachev for introducing the subject of sub-Riemannian geometry and the problem statement. The author is also grateful to Professor B. Bonnard and the Laboratoire de Topologie, Université de Bourgogne, Dijon, France, where this paper was started.

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PROGRAM SYSTEMS INSTITUTE, RUSSIAN ACADEMY OF SCIENCES, 152140 PERESLAVL-ZALESSKY, RUSSIA

E-mail address: sachkov@sys.botik.ru